Optimal Monetary Policy in HANK [*]

Danila Smirnov [†]

June 2023

Abstract

I study the optimal monetary policy in a New Keynesian model with heterogeneous households. The Ramsey planner maximizes aggregate welfare in an economy with rich heterogeneity in the income distribution, as well as a wealth distribution that features an occasionally binding borrowing constraint. I show that heterogeneity qualitatively changes optimal monetary policy relative to the representative agent economy. I highlight that the importance of the novel incentive of the optimal policy in this setting comes from counteracting the increase of hand-to-mouth households in recessions. The mechanism acts through mitigating unequal exposure of households to an aggregate shock, hence the effect is partially present even in the absence of this incentive.

Keywords: heterogeneous agents, borrowing constraint, New Keynesian models.
JEL Codes: E12, E21, E24, E43, E52, E61, G51.

[*]I am grateful to my advisor Davide Debortoli for his invaluable support. I particularly thank Florin Bilbiie, Edouard Challe, Shubhdeep Deb, Jordi Galí, Priit Jeenas, Per Krusell, Albert Marcet, Matthias Meier, Marta Morazzoni, Galo Nuño, Omar Rachedi, Xavier Ragot, Edouard Schaal, Ludwig Straub, Dominik Taller, Henning Weber, and Donghai Zhang for helpful discussions and comments. I would also like to thank participants of CREI Macro Lunch, First PhD Workshop at Sveriges Riksbank, ESEM 2022, and New Challenges to Monetary Policy conference organized by University of Mannheim and EABCN. I acknowledge financial support from the Spanish Ministry of Economy and Competitiveness through the Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000915-S). All errors are my own.

[†]Department of Economics, Universitat Pompeu Fabra. Email: danila.smirnov@upf.edu.

1 Introduction

The normative analysis of the monetary policy was first done in a representative agent framework.[1] The recent rise of attention to inequality has called for a potential reevaluation of the standard price stability prescription. Various approaches were used to identify two main dimensions of heterogeneity that affect the optimal monetary policy:[2] First, the idiosyncratic income risk that leads to cyclical inequality. Second, the binding borrowing constraint that leads to the hand-to-mouth behavior of some households. Both channels call for an optimal monetary policy that departs from strict inflation targeting and sacrifices some of the efficiency in favor of redistribution towards poor or constrained households. The price stability is no longer optimal and monetary authority should be less responsive in adjusting interest rates, allowing for some price volatility.

The majority of this literature was following the common approach of limiting heterogeneity in order to retain analytical tractability. Various simplifying assumptions were utilized: two types of households in fixed proportion, no idiosyncratic risk, or a degenerate distribution of households. Furthermore, recent advancements in numerical solution methods allowed to analyze the optimal monetary policy in models with rich household heterogeneity, and to consider several of the identified channels jointly. Nevertheless, only a few papers exist in this literature and their main focus is on the idiosyncratic risk channel, abstracting from the importance of hand-to-mouth households.

In this paper, following the insights from the "reduced form heterogeneity" literature, I consider a model with nominal price rigidities and rich household heterogeneity that features both the idiosyncratic risk and the occasionally binding borrowing constraint. This latter element implies that, in response to aggregate shocks, the fraction of hand-to-mouth households varies endogenously. Jointly studying both types of heterogeneity allows me to identify a novel channel in the optimal monetary policy that amplifies the differences with respect to the price stability policy derived in the representative agent framework. Amplification comes from the new incentive to reduce this fraction in order to relax the pressure on poor households following a negative aggregate shock.

I consider an Aiyagari-Huggett-İmrohoroğlu incomplete market economy, where households have idiosyncratic labor productivity that drives heterogeneity in individual wealth distribution as well as consumption, labor, and saving decisions. Binding borrowing constraints imply that the bigger the households' debt level, the higher the marginal propensity to consume (MPC). When an exogenous aggregate shock hits the economy, it affects agents differently, depending on their income composition as well as on the nature of the shock. Crucially, households use the credit market to counteract temporary changes in income and to smooth consumption. Those closer or at the borrowing constraint suffer the most since no other mechanism exists for

[1] See Clarida et al. (1999), Woodford (2003), Galí (2015).

[2] See Galí et al. (2004), Bilbiie (2008), Acharya et al. (2021), Nisticó (2016), Le Grand et al. (2021), Bhandari et al. (2021) and a more in-depth discussion in Related Literature section.

intertemporal substitution. The Ramsey planner solves a fully-nonlinear optimization problem in response to unanticipated aggregate shocks.

Constrained households behave in the hand-to-mouth fashion, which means that their consumption is equal to their income. For them, a decline in income caused by an aggregate shock directly translates in a reduction in consumption, disproportionally lowering their utility compared to other households. Previous literature has identified that mitigating this margin calls for partial redistribution, which is achieved by stabilizing the level of output as opposed to stabilizing the level of prices. I show that, if the same negative aggregate shock hits an economy where the fraction of constrained households can vary endogenously, then the share of hand-to-mouth households will increase, calling for even more redistribution.

I calibrate the model to the US economy and validate the calibration by comparing the heterogeneous effects of the monetary policy shock in the model with findings from the empirical literature. I show that an increased importance of the redistribution in light of the new motive that comes from incorporating the occasionally binding borrowing constraint implies substantial quantitative implications for the optimal monetary policy. Specifically, when considering the response of the interest rates to aggregate shocks, the difference between representative and heterogeneous agent frameworks increases by extra 10% to 25%. I illustrate this idea by considering the model specifications representing two alternative ways to characterize heterogeneity in this class of models. Namely, the setting of the two-agent model as in Bilbiie (2008) and the setting absent of the occasionally bonding borrowing constraint as in Bhandari et al. (2021). The mechanism of the optimal monetary policy is always acting through the unequal exposure of households to the monetary policy. Poor households pay interest on debt and rely more on labor income, whereas rich households receive interest on savings and a higher share of firm profits. In response to any aggregate shock that increases inequality, the Ramsey planner has less of an incentive for redistribution in models lacking the varying fraction of constrained households. Hence, the difference with the representative agent model is not as significant as in my model.

Rich heterogeneity and the necessity for a fully nonlinear solution of the dynamic general equilibrium Ramsey problem is a well-known bottleneck in this class of models. I overcome this complication by using a relatively new continuous time approach, as in Kaplan et al. (2018). Combining the analytical part of the solution with the method of first differences allows me to substantially decrease the complexity of the problem and perform the numerical calculations much faster. This method can solve the model very fast without relying on perturbation methods both in the steady state and in the dynamic response to unexpected transitory shocks. I use the optimal control in a dynamic programming setting to solve the optimal policy problem, as in Nuño and Thomas (2022), who nevertheless applied it to a closed economy case. The idea of the method is to use the Lagrangian formulation of the problem in the infinite-horizon setting to find the optimal path for the control variables. I contribute to this literature by developing the calculus of variations solution approach that allows tackling the Ramsey problem in a model with an occasionally binding borrowing constraint.

The rest of the paper is organized as follows: In section 2, I give a more in-depth literature review; in section 3, I describe the model setup; in section 4, I show the Ramsey problem formulation and solution approach; in section 5, I present the results of the calibrated model; and finally section 6 concludes.

2 Related literature

Core of the New Keynesian (NK) literature results as in Clarida et al. (1999), Woodford (2003) and Galí (2015) rely on the assumption of a representative household. Relaxing this assumption was one of the departing points for the recent literature that focused on the model with heterogeneous agents. The addition of various dimensions of heterogeneity can prove to have new valuable insights. However, the main complication restricting advancements in this field was the complexity of such models. Some authors chose to study the models with reduced heterogeneity to avoid numerically heavy solutions and retain the relative simplicity of the analysis.

The crucial dimension of heterogeneity that extended RA models was the introduction of a constant fraction of constrained households, first proposed by Mankiw (2000) to analyze fiscal policy. Later Galí et al. (2004) and Bilbiie (2008) brought this feature into the NK model. These papers show that the presence of constrained agents significantly changes the response of aggregate variables to shocks, the propagation of monetary policy, and the optimal monetary policy under discretion.

Such reduced form modeling of limited asset market participation is beneficial for clarity of analysis, but possibly losing in realism to models with a richer distribution of income and wealth, as in Bewley (1983), Huggett (1993), and Aiyagari (1994) one-asset models. Addressing this concern Debortoli and Galí (2017) illustrated that approximating heterogeneity in the two-agent New Keynesian model (TANK) is not detrimental compared to the models with the complete distribution of heterogeneous agents and nominal rigidities (HANK) for the analysis of monetary policy shocks. In this paper, I show that even though it holds for the propagation of a given policy, the implications for the optimal policy are drastically different.

Extending this class of models Nisticó (2016) introduced transition probability between constrained and unconstrained states. Taking a similar approach, Bilbiie (2019) and Bilbiie (2020) showed the importance of the cyclicality of inequality in an analytical-HANK (a-HANK) model that features two types of households with constant shares, but allows them to switch between the types stochastically. Specifically, the degree to which an income of constrained households moves with respect to aggregate income influences critical predictions of the model. Bilbiie and Ragot (2021) further extends the model by introducing liquidity in the form of money. A working paper by Bilbiie et al. (2020) discusses how the optimal combination of monetary and fiscal policy depends on the cyclicality of inequality relative to the progressivity of the tax. As discussed in these papers, the cyclicality of income inequality plays a crucial role in the behavior of the NK model. Following this literature, I consider the specification of the HANK model with countercyclical inequality.

In the class of two-agent models Challe (2020) can be seen as closely related to my paper because the fraction of constrained households is endogenous and varies with the business cycle. Employment status determines if an agent is constrained or not. Since the fraction of unemployed agents depends on the state of the economy, it opens the door for the monetary policy to affect this margin. The model setup differs from mine since the MPC depends directly on employment without accounting for the wealth effect. Moreover, the income structure is simpler when compared to my model, which makes the optimal policy implications differ from mine. Notably, the idea of policy instruments affecting the share of constrained households plays a crucial role in the outcome of the optimal policy analysis, which is directly related to my idea.

Taking a different approach for attaining analytical tractability Acharya and Dogra (2020) study a Pseudo-RANK model, where the heterogeneity of MPC is omitted, but income risk and precautionary savings are preserved. Acharya et al. (2021) analyze the optimal monetary policy in this setting. Authors show that in the calibration where idiosyncratic income volatility goes up in recessions, to reduce the impact on inequality caused by different exposure to risk, the target criteria should have a higher weight on stabilizing output and smaller weights on output gap and price level stabilization. In line with this literature, I also find that in my richer HANK model, the optimal policy implies a more stable output response with respect to the RANK model.

An alternative approach was used by Le Grand and Ragot (2022), where heterogeneity is restricted by limited history dependencies, which helps to simplify the numerical solution process. The paper studies the optimal unemployment policy over the business cycle. In the context of my paper, the occasionally binding credit constraint that the authors introduce does depend on the aggregate productivity level, so the fraction of constrained agents changes. At the same time, because this transition process is exogenous, the feedback loop from policy to the fraction of constrained agents is missing. Le Grand et al. (2021) are applying the limited history dependence method in a rich HANK setting to show how the combination of monetary and fiscal policies should react to shocks in the economy. The main focus of their paper is showing that monetary policy has no scope for redistribution in the presence of time-varying taxes on labor and capital, as well as highlighting the difference between the timeless and time-0 approach. In contrast, my paper focuses solely on the monetary policy, keeping taxes fixed, and analyses the benefits of reducing the burden of credit constraint by utilizing monetary policy exclusively.

One of the new powerful methods of solving the HANK model involves the approximation around the equilibrium path as shown by Bhandari et al. (2021). These authors study monetary and fiscal policy and find that the Ramsey planner has a strong motive to increase the inflation and labor tax in the short run to redistribute labor income. Higher inflation leads to lower dividend payments and partially reduces the inequality caused by differential income from equity.

Importantly, this holds under the assumption of no binding borrowing constraint, or a constant fraction of constrained households, similar to the TANK literature. I show that in my setting, the model with a natural borrowing limit, as in Bhandari et al. (2021), has significantly lower implications for redistribution when compared to my baseline calibration.

A different approach is taken by McKay and Wolf (2022) where the idea is to capture the incentive of the monetary authority to change its policy in the presence of the heterogeneous agents by finding the weight on the additional term in the quadratic welfare function that captures the change in consumption dispersion. On the one hand, the advantage of this method is in the relative computational simplicity of determining the optimal policy in the presence of heterogeneous agents. On the other hand, and because of the linear approximation, capturing the feedback effect from policy to the fraction of constrained households is impossible.

The solution method that I use in this work makes use of the continuous-time formulation of the New Keynesian model. This approach was developed in works by Kaplan and Violante (2014), Kaplan et al. (2018). In this recent work, authors build the HANK model with two assets to study the effect of an exogenous monetary policy shock on the distribution. They find that in the setting with liquid and illiquid assets, the effect of monetary policy shock on the aggregate outcomes is explained by the indirect effects acting through the distribution of households with varying MPC. This channel outweighs the direct effect that dominates the representative household model.

I build on the method for finding the optimal policy in continuous time models proposed by Dixit and Pindyck (1994) and further developed by Nuño (2017) to calculate the optimal monetary policy in the heterogeneous agents continuous-time model. This method relies on deriving the Lagrangian associated with the optimization problem and calculating its partial derivatives using the calculus of variations. It was used in Nuño and Thomas (2022) in a setting with a small open economy to derive the optimal policy. There was no effect of the changing wealth distribution on aggregate variables, which limited the implications. However, as I will show in this work, the same method can be used in a more complicated setting.

In the recent working paper by González et al. (2022), authors use the method in a New Keynesian model with heterogeneous firms and show how the optimal policy problem can be solved using Dynare. Their solution does not account for the possibility of constraints within the distribution of heterogeneous agents (firms in their case). A similar approach is taken by Dávila and Schaab (2022). The authors analyze the optimal monetary policy in the HANK setting. As in González et al. (2022), before solving the Ramsey problem, authors discretize it to show how to properly take into account the borrowing constraint faced by the households. Dávila and Schaab (2022) call this procedure the "primal approach" and apply it to use the solution method with the sequence-space Jacobians. Compared to my method, I don't rely on a discretization, as all the properties are derived directly in the continuous time formulation of the Ramsey problem. Avoiding constraints on the numerical tools used to solve the differential equations is an advantage of my approach.

Moreover, the focus of this paper is different, highlighting the importance of the feedback loop from the policy to the fraction of constrained households. I argue that the HANK model with an occasionally binding borrowing constraint has significantly different policy implications compared to the models previously used in the literature. This comparative analysis also distinguishes my work from Dávila and Schaab (2022).

3 Model

In this section, I introduce the model used in the subsequent analysis. It is a standard New Keynesian model in continuous time[3] with monopolistic competition in the intermediate goods market with sticky prices and no aggregate uncertainty. The economy's supply side and the government block are kept at a bare minimum to keep the model as tractable as possible. The household block is the one that has most of the complexity and the one that crucially affects the way the optimal monetary policy is conducted. Households decide on their consumption, labor supply, and investment in nominal bonds, given the aggregate values in the economy, household-specific productivity, and bond holdings of this household. This decision is subject to the resource constraint, and the binding borrowing constraint. Below I describe the model in more detail.

3.1 Household side

There is a continuum of infinitely-lived households indexed by $i \in [0, 1]$ that have a unit mass and differ in their bond holdings, labor productivity, and equity shares. All households have the same discounting factor ρ. They maximize their expected utility over the infinite lifetime span subject to the budget constraint, choosing a consumption path, a labor supply path, and a bond investment path. Households face a borrowing constraint $b_{i,t} \geq \underline{b} < 0$

$$\mathcal{V}_{i,t} = \max_{\{c_{i,t}, l_{i,t}, \dot{b}_{i,t}\}_t} \mathbb{E}_t \int_0^{+\infty} e^{-\rho t} \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} \right) dt \tag{1}$$

$$\text{s.t.} \quad c_{i,t} + \dot{b}_{i,t} = \lambda W_t l_{i,t} \varepsilon_{i,t} + d_t \varepsilon_{i,t} - T_t + r_t^b b_{i,t} \tag{2}$$

$$b_{i,t} \geq \underline{b} \tag{3}$$

Where $b_{i,t}$ are bond holdings that are priced in nominal terms, thus their value depreciates at an inflation rate[4]. Real interest rate equals nominal interest rate less inflation $r_t^b = i_t - \pi_t$.

I follow the standard assumption in the literature that the borrowing constraint is constant

[3] Using continuous time allows to simplify some of the calculations, such as deriving the law of motion for the households distribution. Also, it increases the efficiency of the optimal policy derivation. More on the advantages of continuous time approach later.

[4] In the rest of the paper, I treat all the variables in real terms. This also can be seen as the central bank effectively controlling the real rate directly, which lets me omit the step with the Fischer equation for simplicity of exposition. This simplification does not affect the indeterminacy of price level in the model and still calls for the central bank policy to determine the equilibrium.

across households and time[5]. Presence of the binding borrowing constraint is going to play a crucial role in the analysis of the Ramsey problem solution. One of the margins of the optimal monetary policy will be to relax this constraint, to allow more households to take an advantage of financial markets to smooth their consumption path in response to aggregate shocks.

The idiosyncratic productivity level $\varepsilon_{i,t}$ is the exponent of $e_{i,t}$ that follows the Ornstein-Uhlenbeck process with zero mean, a mean-reversion parameter ρ_e, and a Wiener process $dW_{e,i,t}$ multiplied by a parameter for the standard deviation of the shock σ_e

$$\varepsilon_{i,t} = exp\{e_{i,t}\};\ de_{i,t} = -\rho_e e_{i,t} dt + \sigma_e dW_{e,i,t} \tag{4}$$

Households receive a labor subsidy λ that cancels the inefficiency associated with monopolistic competition and implies a negative lump-sum transfer T_t to balance the government budget that finances the labor subsidy. Finally, households receive dividends from firms that are distributed proportionally to the household productivity level.[6]

A distribution of households over the space of bond holdings and productivity levels is denoted by f_t. This distribution is a two-dimensional object that is constant in the steady state, or follows a deterministic path in transition, due to the law of large numbers. Notably, the evolution associated with the idiosyncratic shock is determined exogenously by the law of motion of the Ornstein-Uhlenbeck process. In contrast, the evolution of bond holdings is determined by the optimizing behavior of households.

The Hamilton-Jacobi-Bellman (HJB) equation for the household problem takes the following form

$$\rho \mathcal{V}_{i,t} = \max_{c_{i,t}, l_{i,t}} \left\{ \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \left(\lambda W_t l_{i,t} \varepsilon_{i,t} + d_t \varepsilon_{i,t} + T_t + r_t^b b_{i,t} - c_{i,t} \right) \frac{\partial \mathcal{V}}{\partial b} \right\} - \tag{5}$$

$$- \varepsilon_{i,t} \rho_e e_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_e^2}{2} \frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} \tag{6}$$

Where the first two terms represent an instantaneous utility, the third term governs the change of the value function because of the change of bond holdings, the following two terms correspond to the change associated with the stochastic labor productivity process, and the last term is capturing changes associated with aggregate variables movements.

Solving this maximization problem for the optimal value of consumption and labor, and combining it with the HJB equation, I get optimality conditions for the household problem in a form of the partial differential equation, which is a standard step in the continuous-time

[5] A model with an endogenous bank's decision on the borrowing constraint can have different implications. Finding if the borrowing limit that depends on the economy's aggregate state is a significant margin or not, I leave it for future research.

[6] I also check that results are robust to the model specification with the lump-sum distribution of dividends.

literature.[7]

$$c_{i,t} = \left(\frac{\partial \mathcal{V}_{i,t}}{\partial b}\right)^{-\frac{1}{\nu}} \tag{7}$$

$$l_{i,t} = \left(\frac{\lambda W_t \varepsilon_{i,t}}{\varphi}\frac{\partial \mathcal{V}_{i,t}}{\partial b}\right)^{\frac{1}{\gamma}} \tag{8}$$

$$\rho \mathcal{V}_{i,t} = \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t}\mathcal{V}_{i,t} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} \tag{9}$$

Where $\mathcal{A}_{i,t}$ is an infinitesimal generator of process $\mathcal{V}_{i,t}$, that captures changes in the value function due to the evolution of household-specific state variables $\varepsilon_{i,t}$ and $b_{i,t}$

$$\mathcal{A}_{i,t}\mathcal{V}_{i,t} = \left(\lambda W_t l_{i,t}\varepsilon_{i,t} + d_t \varepsilon_{i,t} + T_t + r_t^b b_{i,t} - c_{i,t}\right)\frac{\partial \mathcal{V}}{\partial b} - \varepsilon_{i,t}\rho_e e_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t}\frac{\sigma_e^2}{2}\frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} \tag{10}$$

3.2 Fokker–Planck / Kolmogorov forward equation

The law of motion of the joint distribution of bond holdings and labor productivities can be characterized by Fokker–Planck or Kolmogorov forward equation

$$\frac{\partial f_{i,t}}{\partial t} = -\frac{\partial}{\partial b}\left[\left(\lambda W_t l_{i,t}\varepsilon_{i,t} + d_t\varepsilon_{i,t} + T_t + r_t^b b_{i,t} - c_{i,t}\right) f_{i,t}\right] + \tag{11}$$

$$+ \frac{\partial}{\partial \varepsilon}\left[\varepsilon_{i,t}\rho_e e_{i,t} f_{i,t}\right] + \frac{\partial^2}{\partial \varepsilon^2}\left[\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2} f_{i,t}\right] = \mathcal{A}_{i,t}^* f_{i,t} \tag{12}$$

Notice that conceptually an evolution of the distribution should be tightly linked to the household problem. Borrowing/saving decisions of households change the distribution of bonds in the economy. For a given borrowing/saving decision on the space of labor productivity and bond holdings, there is a unique distribution of households that stays constant. For a given distribution, borrowing/saving decisions determine its evolution.

Mathematically speaking, a law of motion of the household value function adjoins a law of motion of distribution. See Appendix for the analytical derivation of this fact in the setting of my model. For the standard Aiyagari–Bewley–Huggett model, see the paper by Achdou et al. (2021).

3.3 Supply side

The final good is produced by competitive firms that use the continuum of intermediate goods in the production process. A single producer supplies each intermediate good. Intermediate producers have a unit mass and are indexed by $j \in [0, 1]$. The production function of the final good producers has a CES structure with ϕ_t determining the price elasticity of demand for the

[7] See paper by Achdou et al. (2021) for the extensive explanation of the solution method of consumption-saving problem in continuous time.

intermediate goods $y_{j,t}$.

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\phi_t - 1}{\phi_t}} dj \right)^{\frac{\phi_t}{\phi_t - 1}} \tag{13}$$

Producers of the final good are competitive and take both the price P_t of the final good and the prices of intermediate goods $p_{j,t}$ as given. Thus they maximize the following expression, subject to the production function above

$$\max_{\{y_{j,t}\}_{j \in [0,1]}} P_t Y_t - \int_0^1 p_{j,t} y_{j,t} dj \tag{14}$$

The outcome of the profit maximization is a demand function

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t} \right)^{-\phi_t} Y_t \tag{15}$$

Zero profit condition arises from the assumption of CES demand for intermediate goods and perfect competition in the final good market. This condition, together with the previous optimization result, yields:

$$P_t = \left(\int_0^1 p_{j,t}^{1-\phi_t} dj \right)^{\frac{1}{1-\phi_t}} \tag{16}$$

Having derived the demand function facing the intermediate producers, we can now turn to their problem. These producers are monopolists with a linear production function that is the same for all $j \in [0, 1]$. For simplicity, I assume that a given firm j in its production process uses only labor $n_{j,t}$ as input and features linear production technology.

$$y_{j,t} = \theta_t n_{j,t} \tag{17}$$

The aggregate productivity level θ_t is common across all firms and equals 1 in the steady state. When the TFP shock arrives, θ_t temporarily deviates from the steady state level with gradual convergence back to the original level.

Intermediate goods producers set prices optimally given the demand functions from the final good producers, and competing monopolistically. Thus the j-th firm maximization problem is

$$\max_{\{\dot{p}_{j,t}\}_t} \int_0^\infty e^{-\int_0^t r_\tau^b d\tau} \left[\left(\frac{p_{jt}}{P_t} - m_t \right) \left(\frac{p_{j,t}}{P_t} \right)^{-\phi_t} Y_t - \frac{\psi}{2} \left(\frac{\dot{p}_{j,t}}{p_{j,t}} \right)^2 Y_t \right] dt \tag{18}$$

$m_t = \dfrac{W_t}{\theta_t}$ is the real marginal cost of production, with W_t denoting the real wage at time period t and $-\int_0^t r_\tau^b d\tau$ is the discount factor[8].

[8]Notice, that the firms' discounting factor is equal to the return on the bonds market. This relies on the

For a firm j at time t the value function is denoted by $\mathcal{J}_{j,t}$. The Hamilton-Jacobi-Bellman equation for the above problem has the following form

$$r_t^b \mathcal{J}_{j,t} = \max_{\dot{p}_{j,t}} \left(\frac{p_{j,t}}{P_t} - m_t\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\phi_t} Y_t - \frac{\psi}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t + \dot{p}_{j,t}\frac{\partial \mathcal{J}_t}{\partial p} + \frac{\partial \mathcal{J}_t}{\partial t} \tag{19}$$

Taking derivative with respect to inflation $\pi_{j,t} = \frac{\dot{p}_{j,t}}{p_{j,t}}$ (maximization problem) and price $p_{j,t}$ (using the Envelope theorem) and using the symmetry of solutions of all firms in the equilibrium I get the Phillips curve that links inflation with the markup gap in the economy

$$\left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1} m_t - 1\right) + \dot{\pi}_t \tag{20}$$

Moreover, I assume that intermediate firms are owned by a fund that distributes profits among all the households in the economy, proportional to their labor productivity[9]. Since both intermediate firms as well as households have unit masses, dividends that go to households divided by the average labor productivity are equal to firms' aggregate profits

$$D_t = \left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t - W_t N_t \tag{21}$$

Finally, total output is equal to the representative firm output in the symmetric equilibrium also because of the unit mass of firms and because they are identical. This means as well that the labor demand of each firm is equal to the aggregate labor demand $n_{j,t} = N_t$

$$Y_t = y_{j,t} = \theta_t n_{j,t} = \theta_t N_t \tag{22}$$

3.4 Markets clearing

In addition to the supply side and the household side, there is also the government that only provides the labor subsidy, as mentioned before. The subsidy λ to workers is such that it cancels out the inefficiency created by the monopolistic competition of intermediate goods producers. The government runs a balanced budget and provides the lump-sum transfer T

$$\lambda = \frac{\phi}{\phi - 1}, \quad T_t = -\left\langle (1-\lambda) W_t l_{i,t}\varepsilon_{i,t}, f_{i,t}\right\rangle = -\int_i (1-\lambda) W_t l_{i,t}\varepsilon_{i,t} f_{i,t} di$$

Where $\langle \cdot, \cdot \rangle$ stands for the integration over the distribution, see a more detailed explanation in Appendix. There are no additional redistributions done by the government apart from the one described above.

assumption that firms are owned by the mutual fund.

[9] I also check that results are robust to the alternative assumption of the lump-sum distribution of profits.

In the equilibrium the bond market clears to a zero aggregate bond supply.

$$\langle b_{i,t}, f_{i,t} \rangle = 0$$

The labor market clears with a wage W_t so that a labor demand equals to an effective labor supply

$$\frac{Y_t}{\theta_t} = N_t = \langle \varepsilon_{i,t} l_{i,t}, f_{i,t} \rangle$$

Dividends paid to households are equal to profits of intermediate firms

$$\langle d_t \varepsilon_{i,t}, f_{i,t} \rangle = D_t = \left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t - W_t \langle l_{i,t}\varepsilon_{i,t}, f_{i,t} \rangle$$

Which implies

$$d_t = \left[\left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t - W_t \langle l_{i,t}\varepsilon_{i,t}, f_{i,t} \rangle\right] \frac{1}{\overline{\varepsilon}}$$

Where $\overline{\varepsilon} = \langle \varepsilon_{i,t}, f_{i,t} \rangle$ is an average labor productivity.

Finally, the last equation that determines the equilibrium is the goods market clearing condition

$$C_t = \left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t$$

Notice that one equation is still missing for the determinacy of the equilibrium. This is the equation that defines a monetary policy. Since the policy has to be optimal, it is determined by optimality conditions of the Ramsey planner problem discussed in the next section.

4 Ramsey problem

I study a transition path under the full commitment Ramsey optimal policy in response to the one-time, unanticipated, transitory aggregate shock. The Ramsey problem is a maximization of the utilitarian value function subject to all equations that define the competitive equilibrium

$$\mathcal{V}_{\mathcal{G}} = \max_{f,\mathcal{V},c,l,W,Y,\pi,r^b} \int_0^1 \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} \right) f_{i,t} dt di \tag{23}$$

$$\text{s.t.} \{\text{competitive equilibrium equations}\} \tag{24}$$

Because of the law of large numbers and no aggregate uncertainty, the expectation can be simplified. I solve the Ramsey optimal monetary policy by the Lagrangian method. The difficulty in this application is that all objects with respect to which derivatives have to be taken are functions, not variables. Partially this is because of the combination of transition dynamics with a continuous-time approach. Taking a derivative with respect to a control variable is done not period-by-period as in a discrete time approach but rather on all path simultaneously. This is not a real complication because, conceptually, it is identical to a discrete time method, and intuitions follow the same logic.[10] At the same time, some objects are functions even within one period. For example, the distribution of households is a two-dimensional object at every instant of time, which means that a different approach to taking derivatives has to be pursued.[11]

Nuño and Thomas (2022) proposed a method for obtaining the Ramsey policy in continuous time with heterogeneous agents. Their method involves writing the social planner problem in a form of Lagrangian and taking Gateaux derivatives with respect to state variables. In this paper, I slightly change the methodology using Calculus of Variations instead of Gateaux differentials. On the one hand, the method is very similar in logic, so main results are unchanged. On the other hand, using the calculus of variations is much better suited for analyzing the problem's behavior at the boundary constraints. It can be done with Gateaux differentials in most straightforward cases, but as soon as constraints' conditions are more complex, which is the case for the binding borrowing constraint, it cannot be used conveniently.

A simple intuition behind the advantage of using the calculus of variation in this problem is in restricting the space of policy functions within which the Ramsey planner has to search for the optimal policy. Careful consideration of the binding borrowing constraint when solving for the optimal policy already restricts the space of available policies to those that satisfy the binding borrowing limit. In other words, this new method will never propose a policy for which some households will violate the borrowing constraint. This dramatically reduces the difficulty of the numerical solution since the guess and verify step for an appropriate policy respecting the binding borrowing limit is omitted[12].

I apply this method to solve for the Ramsey optimal policy in a general equilibrium setting of the HANK model described above and show how to perform calculations to get a system of differential equations that can be solved numerically and that define the optimal policy in this model. The method proves to be very convenient, as the resulting system of differential equations for the co-states is not just linear, which is valid for any Lagrangian problem, but also has a symmetry with respect to the original problem. See derivations and a more detailed

[10]In the Appendix, I include the derivation of the Ramsey problem for the standard RANK and as the TANK model in continuous time. These two models serve as simpler examples for the derivations and reference models for comparing the results.

[11]Distribution of households is not the only function with respect to which Ramsey planner has to maximize welfare. Other two-dimensional objects include consumption, labor, and household value function. The exact derivation of the Ramsey problem can be found in the Appendix.

[12]The logic behind this method is similar to the endogenous grid method Carroll (2006) used in some quantitative macro models. Calculus of variations is widely used in other fields of economics, e.g., a wide range of continuous-time applications in finance can be found in Malliavin and Thalmaier (2006) and Clarke (2013).

intuition in Appendix.

$$
\begin{aligned}
&\mathcal{L}[f,\mathcal{V},c,l,W,Y,\pi,r^b,T,d] = && (25)\\
&= \int_0^\infty e^{-\rho t}\left[\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma}, f_{i,t}\right\rangle + \left\langle \zeta_{i,t}, \mathcal{A}^*_{i,t} f_{i,t} - \frac{\partial f_{i,t}}{\partial t}\right\rangle\right. && (26)\\
&\quad + \left\langle \varrho_{i,t}, \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t}\mathcal{V}_{i,t} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} - \rho\mathcal{V}_{i,t}\right\rangle && (27)\\
&\quad + \left\langle \mu_{i,t}, c_{i,t}^{-\nu} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}\right\rangle + \left\langle \kappa_{i,t}, l_{i,t}^{\gamma} c_{i,t}^{\nu} - \frac{\lambda W_t \varepsilon_{i,t}}{\varphi}\right\rangle && (28)\\
&\quad + \eta_{b,t}\langle b_{i,t}, f_{i,t}\rangle + \eta_{Y,t}\left(Y_t - \theta_t\langle l_{i,t}\varepsilon_{i,t}, f_{i,t}\rangle\right) && (29)\\
&\quad + \eta_{T,t}\left(T_t - (1-\lambda)\frac{W_t}{\theta_t}Y_t\right) + \eta_{d,t}\left(d_t - \left(1 - \frac{\psi}{2}\pi_t^2 - \frac{W_t}{\theta_t}\right)\frac{Y_t}{\overline{\varepsilon}}\right) && (30)\\
&\quad \left. + \eta_{\pi,t}\left(\frac{\phi-1}{\psi}\left(\frac{\phi}{\phi-1}\frac{W_t}{\theta_t} - 1\right) + \dot{\pi}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t\right)\right] dt && (31)
\end{aligned}
$$

Where

$$
\begin{aligned}
\mathcal{A}_{i,t}\mathcal{V}_{i,t} =& \dot{b}_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial b} - \rho_\varepsilon \varepsilon_{i,t} e_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} && (32)\\
\mathcal{A}^*_{i,t} f_{i,t} =& -\frac{\partial}{\partial b}\dot{b}_{i,t} f_{i,t} + \frac{\partial}{\partial \varepsilon}\rho_\varepsilon \varepsilon_{i,t} e_{i,t} f_{i,t} + \frac{\partial^2}{\partial \varepsilon^2}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2} f_{i,t} && (33)\\
\dot{b}_{i,t} =& \lambda W_t l_{i,t}\varepsilon_{i,t} + r_t^b b_{i,t} + T_t + d_t\varepsilon_{i,t} - c_{i,t} && (34)
\end{aligned}
$$

Notation $\langle \cdot, \cdot \rangle$ represents an inner product of two functions, defined on a common domain D. In this case, it is the space of productivity levels ε and bond holdings b

$$
\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma}, f_{i,t}\right\rangle = \int_{\underline{b}}^\infty \int_0^\infty \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma}\right) f_{i,t}\; d\varepsilon_{i,t} db_{i,t} \tag{35}
$$

In Appendix, I show how to take differentials with respect to control variables using calculus of variations and how to rearrange equations to get equations for an updating solution algorithm.

One of the important features of this model is the time inconsistency of the Ramsey commitment policy. After an initial maximization, if the planner has an opportunity to reoptimize, the policy will change with respect to the one the planner initially committed to. This is the case both in the presence of the MIT shocks and without them. Solving for the optimal Ramsey policy without MIT shock reveals a strong incentive for the redistribution from lenders to borrowers that is achieved through raising inflation immediately after the implementation of the Ramsey policy. After the initial redistribution, the economy gradually reverts to the optimal long-run steady state with constant inflation.[13] This is what the literature refers to as the *time-0* approach. In a setting with heterogeneous agents it is studied in papers Bhandari

[13]The long-run inflation for the calibrated model is not economically different from zero in my model. The value is between -0.1% and 0.1%, depending on the assumption on the distribution of profits.

et al. (2021), Acharya et al. (2021), Le Grand and Ragot (2022), Nuño and Thomas (2022), and Dávila and Schaab (2022). My model delivers the same prediction as other papers for the strong redistributive incentive that dwarfs any response to relatively small MIT shocks.

In this paper, I concentrate my attention on studying the optimal Ramsey policy from what is called in the literature the *timeless perspective* (see Woodford (2003)). Essentially this means that a starting point for the Ramsey planner is determined by the steady state distribution of households and past commitments of the Ramsey planner that I take from the steady state as well. This is a meaningfull initialization for past commitments, since in this model the steady state commitments do not depend on the path that was leading to the steady state. In the continuous-time setting, this implies that paths of the inflation and value functions of households have to be such that forward-looking co-state variables start at steady state values. In other words, on impact of the aggregate shock, the Ramsey planner effectively keeps past commitments and gets to choose the policy only on the subset of available responses, which is constrained by the necessity to satisfy promises from before an unexpected aggregate shock.[14]

4.1 Solution algorithm

The solution algorithm aims to find the equilibrium response under the optimal monetary policy to an unexpected transitory shock (MIT shock). I study a negative TFP and a shock to the demand elasticity that lowers the desired markup. After an arrival of the unexpected shock, agents have the perfect foresight of changes in aggregate variables from that moment onward. For households, this is captured by the $\frac{\partial \mathcal{V}}{\partial t}$. This term summarizes changes in a households' value due to changes in aggregate variables.

First, I conjecture some path for aggregate variables that enter the household problem in response to a shock. Specifically, fix a path for the real interest rate r^b_t.[15] Given this path, solve the household problem. Starting at time T, very distant from the shock impact, allows assuming that the household value at this point coincides with the one in the steady state so that I can solve for the value of $\mathcal{V}_T$, given state variables at that point. Then, using backward induction, solve for the path of $\mathcal{V}_t$ back to the initial impact at $t = 0$. Having solved the household choices for the whole path, one can now use forward induction to determine a corresponding path of the distribution. Specifically, starting at $t = 0$ and taking the distribution f_0 to be the one from the steady state, use the Fokker–Planck / Kolmogorov forward equation to solve for a path of the distribution up to time T. Having determined the distribution path, check if all markets clear. If they do not, update corresponding prices and aggregate variables (notice that there is no update for the policy yet) and start the procedure again.

[14] Mathematically this means that in the Lagrangian problem I have to include two additional constraints on the initial level of inflation and household value function, with the co-state variables equal to the initial co-state variables on the forward-looking equations. See more detail in the Appendix.

[15] As mentioned above, the real interest rate has to be considered the monetary policy instrument. This allows omitting the Fisher equation and thinking directly in terms of the real variables in the economy. At the same time, this does not create indeterminacy because the "missing" equation is still there, and one can determine the nominal interest rate using the Fisher equation if needed.

Second, I find an update for the monetary policy. For this update, solve the corresponding costate variables problem from Lagrangian first-order conditions. Since these equations have the form of linear differential equations, this step is easier than the first step. Specifically, guess paths of costate variables, in the same way, as paths of state variables, by initializing them at steady state values. Then start at the time T, assuming that the steady state has been attained by this time. Solve for stationary values of multipliers associated with the first-order condition on the distribution. Then, following the analogy with the value function, use backward induction to solve backward for a path of this costate two-dimensional object until $t = 0$. Then take costates associated with the value function and its values to be the same at the moment of impact as in the steady state. Notice that this object is two-dimensional as well. Also, the assumption that starting values are from the steady state has to do with the fact that I am solving the problem under *timeless perspective*. Specifically, if the Ramsey planner could ignore previous commitments as soon as a shock hits, the value for this costate should have been zero, as discussed above. Following an analogy with the distribution, solve forward for values of this costate into the future until time T. Then, update a guess for costate variables using the rest of first-order conditions. Finally, as in every Lagrangian problem, an extra equation left allows one to get an update for the policy itself. Repeat the process for costate variables until the convergence and use the final update on the policy function to change it and go to the first step again.

The solution is achieved when all residuals from equilibrium and optimality conditions are smaller than a predefined convergence criterion. Notice that first and second steps that solve for state and costate variables equilibria do not have to be done separately. Updates can be done simultaneously, giving a faster convergence speed.

4.2 Calibration

To calibrate the model, I use parameters from two papers closest to mine in the application. Specifically, for the supply side and household preference, I take parameters from Kaplan et al. (2018), and for the labor income process and inequality, I take moments from Bhandari et al. (2021). Final good producers have a price elasticity parameter $\phi = 10$, which implies markups for intermediate firms are equal to around 10%, and a cost of price adjustment is $\psi = 100$. This means that the resulting slope of the Phillips curve is $\phi/\psi = 0.1$. Notice that in this setting, the actual markup of firms is not that important. Specifically, it governs profits and, thus, the relative share of dividends in the income of households. However, dividends are distributed proportionally to labor productivity similarly to the labor income, so the impact of potential redistribution from wages to dividends is mild.

Household utility function has a risk aversion of $\nu = 1$, and an inverse Frisch elasticity of labor supply $\gamma = 1$. The household discount factor is calibrated to match a yearly real interest rate of 3%. The borrowing limit is calibrated to deliver 30% of constrained households in a steady state. The value of $\underline{b}$ for which this is achieved is approximately equal to 1.4

average yearly labor income in the population and 3.8 average yearly labor income of constrained households. Finally, to calibrate two parameters for the labor productivity process, I match the variance of a yearly labor income change in one year (for the variance of the stochastic component) and the variance of the natural logarithm of labor earnings in the cross-section (for the mean reversion parameter). All of parameters and their values can be seen in Table 1[16].

Fixed	**Description**	**Value**		
ν	Risk aversion	1		
$1/\gamma$	Frisch elasticity of labor supply	1		
ϕ	Price elasticity of demand	10	(slope of the Phillips Curve	
ψ	Price adjustment cost	100	$\phi/\psi = 0.1$)	
Fitted	**Description**	**Value**	**Moment**	**Value**
ρ	Discount rate	0.118	real return	3%
$\underline{b}$	Borrowing limit	-1.23	% constrained	30%
ρ_e	Mean reversion	0.11	var $log(LI)$	0.7
σ_e	Volatility	0.39	var $\Delta(LI)$	0.23

Table 1: Main calibration for HANK

When performing a comparison with the representative agent model (RANK), the two-agent model (TANK), and the model with heterogeneous households that can borrow up to the natural borrowing limit (NB HANK), I recalibrate parameters accordingly. Specifically, in the RANK model, I change the discount rate ρ to maintain the steady state level of interest rate $r^b = 3\%$. In the TANK model, I take the fraction of constrained agents to be 30%, discount factor $\rho = r^b = 3\%$, and a labor productivity of constrained agents to match the average labor productivity of constrained agents in the HANK model. Finally, for the NB HANK model, I recalibrate ρ to maintain the equilibrium interest rate $r^b = 3\%$, and ρ_e to maintain the same level of the variance of the crossectional distribution of log labor income. See recalibrated parameter values for TANK and NB HANK models in Table 2.

5 Results

5.1 Steady state

Solving the model in the steady state and finding the optimal long-run level of inflation reveals that the optimal inflation is not significantly different from zero. Since all agents perfectly anticipate steady state inflation, the only reason for the inflation to be non-zero is to change the balance between dividends and wages. Households also anticipate this redistribution mechanism. The main effect of a non-zero steady state inflation will be higher or lower demand for bonds

[16] I assume there is no ZLB in this setup and solve the model under this assumption.

	Description	Value	Moment	Value
RANK				
ρ	Discount rate	0.03	real return	3%
ε	Labor productivity	1		
TANK				
ρ	Discount rate	0.03	real return	3%
$\underline{b}$	Borrowing limit	-1.23	HANK model	-1.23
f_k	Share of constrained	0.3	HANK model	0.3
e_k	log of labor productivity of constrained	-0.86	average labor productivity of constrained in HANK	0.42
NB HANK				
ρ	Discount rate	0.031	real return	3%
$\underline{b}$	Borrowing limit	-100	% constrained	0.6%
ρ_e	Mean reversion	0.69	var $log(LI)$	0.7
σ_e	Volatility	0.39	HANK model	0.39

Table 2: Calibration for RANK, TANK, and NB HANK

to compensate for changes in the labor income and the dividend composition. This means that by having a non-zero inflation, the Ramsey planner has to accept a permanent cost of inflation for almost no real effects since the potential anticipated redistribution is dampened by an endogenous change in the bond holdings distribution. In the calibrated model, the optimal steady state level of inflation is less than 0.1%.

As discussed in the calibration section, the steady state is characterized by the distribution of bonds with a unit mass of 0.3 at $\underline{b}$. The rest of the distribution is shown in Figure 1. I also show graphs illustrating consumption and saving decisions and the income composition. For the readability of plots, I display average values for a given level of bond holdings (on the x-axis) to reduce the dimensionality of plots. The 3D plots on the space of bond holdings and labor productivities can be found in Appendix.

Households choose their consumption and labor supply optimally, depending on their current productivity level, bond holdings, and aggregate state variables. The steady state is characterized by following facts that will be important for the unequal exposure to aggregate shocks through income. First, households with lower bond holdings consume less and supply more labor in terms of hours worked. At the same time, because of the endogenous positive correlation of bond holdings and labor productivities, the effective labor supply measured in efficiency units is increasing in bond holdings, even though the labor supply decreases. The dividend income also increases with the bond holdings because of the assumption that dividends are distributed proportionally to the labor productivity.[17] Transfers are lump-sum, so the level does not change for various bond holdings.

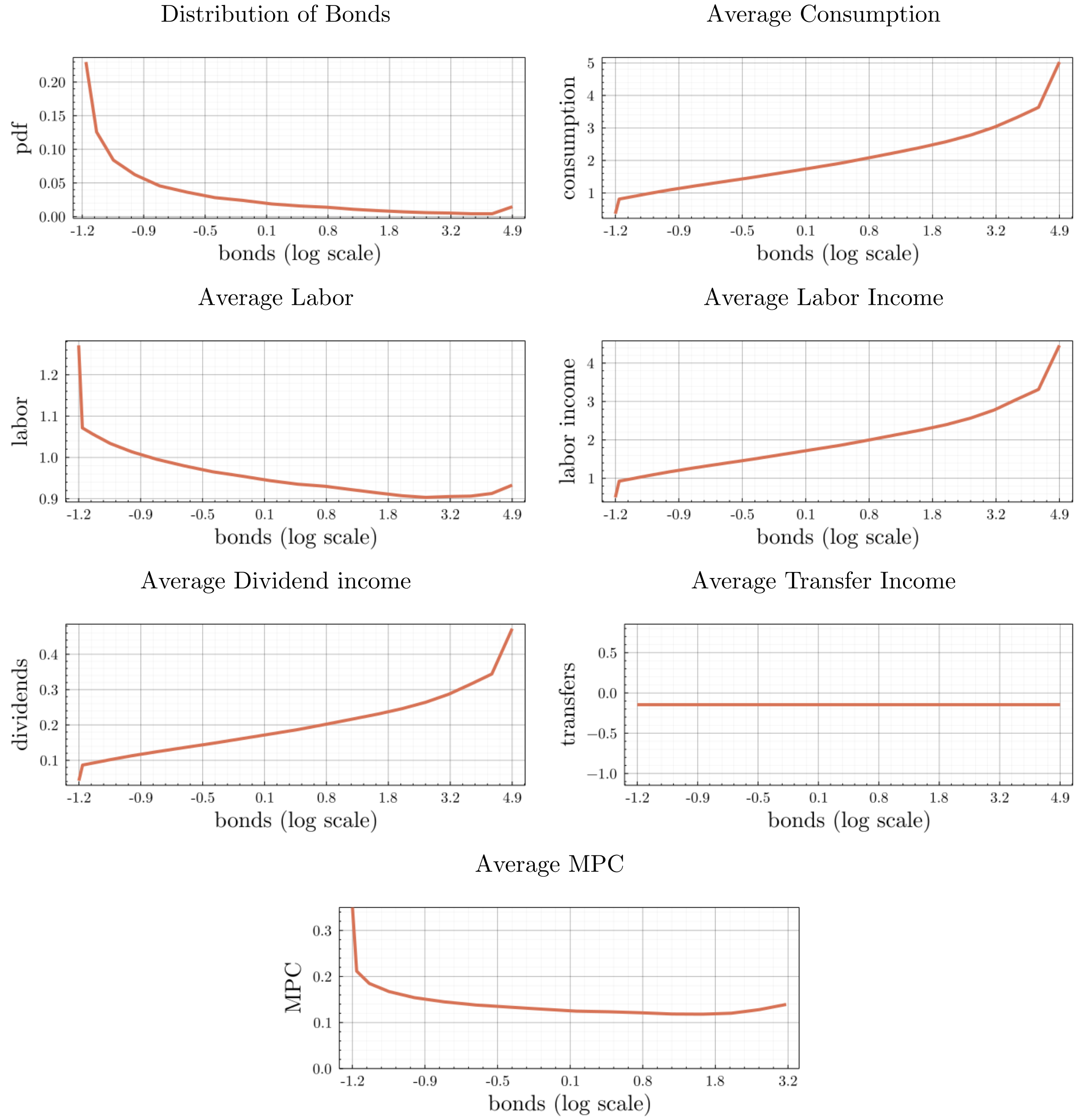


Figure 1: Steady state

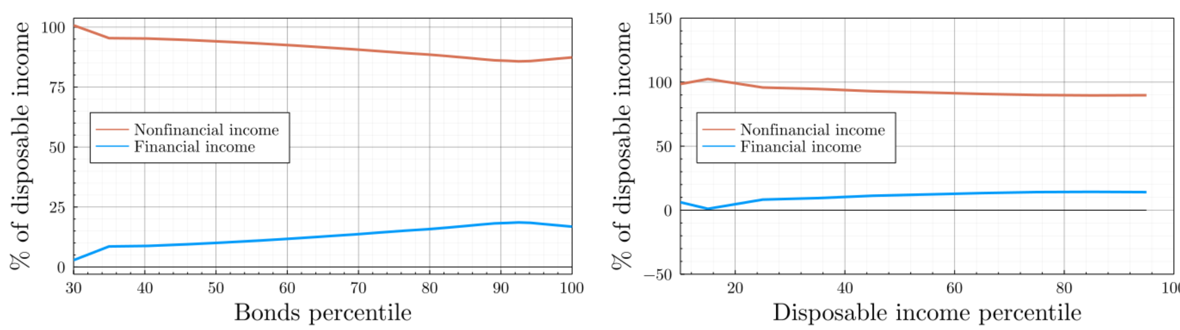
Another important feature that will play a crucial role in analyzing dynamic responses to shocks can already be seen in steady state plots. Households on the boundaries of the graph display a visible discontinuity compared to those just short of the constraints. For the top of the bond distribution this is attributed to the truncation error. This numerical artifact is insignificant because of the small fraction of households at that point. However, it is an important feature of the model at the borrowing constraint $\underline{b}$. On the one hand, this difference comes from the aggregation of households for the productivity values that result in the point

[17]This changes depending on the assumption about the distribution rules of dividends. I check that the main results are robust to the case with the lump-sum distribution of dividends.

mass. The 3D plots for household policy functions do not display such discontinuity. On the other hand, this is what the two-agent New Keynesian literature highlights by modeling the difference between Ricardian and non-Ricardian agents. In this model, the transition to a fully non-Ricardian behavior on the constraint is gradual because of the continuum of households. Hence, the households drastically change their marginal propensity to consume for the relatively small changes in wealth close to the constraint.

Composition of household income components in the steady state plays a crucial role in determining the propagation of monetary policy shocks. Recent empirical evidence by Holm et al. (2021), Andersen et al. (2020), Flodén et al. (2020), and Amberg et al. (2022) shows that propagation of monetary policy shocks is largely driven by the direct effect of changing interest rates on income, and not by the intertemporal shift of consumption. Explanation of this empirical fact relies on the differential exposure of households to interest rates. Some are borrowers and receive a negative income shock when rates increase. Some have a positive net asset position and benefit from higher rates. This dimension of heterogeneity of exposure to interest rates is at the core of the monetary policy transmission, as suggested by the empirical evidence. Since the key dimension of heterogeneity in my model is a bond position, the same mechanism also plays a significant role in my analysis. In order to have an understanding of similarities and differences between my model and empirical evidence, I compare the model's behavior with empirical facts in the steady state and later in response to a monetary policy shock.

Starting with the steady state analysis, the most direct comparison is with Holm et al. (2021), which uses data from Norway to capture differential effects of the monetary policy, conditional on liquidity holdings, which is precisely the dimension of heterogeneity in my model. In Figure 2a, I plot two separate sources of income for households in the model, conditional on their bond holdings: financial income is the return on holding bonds, and income from dividends and non-financial income is everything else. I find that the importance of financial income increases with higher liquidity and non-financial income share decreases, which is in line with the empirical evidence.



(a) conditional on wealth

(b) conditional on income

Figure 2: Cash flow shares

It is more difficult to compare my model with Andersen et al. (2020). The caveat is that the dimension of heterogeneity in the paper is the household's disposable income. Since households with higher disposable income hold, on average more debt, my model cannot be directly compared, as I do not have wealthy hand-to-mouth agents driving this empirical result. Moreover, since my model does not have any redistributive taxation, I have to perform an aggregation of income into financial and non-financial categories. Despite these shortcomings, values in the model have similar behavior to their counterparts in the data. Specifically, when looking at Figure 2b, I conclude that in my model, households with higher income have relatively lower importance of non-financial income, which is in line with the data.

Finally, comparing with the empirical evidence from Flodén et al. (2020), I conclude that in the same way, as documented in the paper based on the data from Sweden, households with higher debt to income ratio have higher debt expenditures shares. Empirically, the cause of a higher debt position of these households is the debt taken for the purchase of a house. Even though my model does not have housing, the primary implication of a negative effect of increasing interest rates on indebted households is captured by my model. It plays a crucial role in implications for the optimal monetary policy.

5.2 Dynamic response to the contractionary Monetary Policy shock under the simple rule

Before looking at the optimal monetary policy responses to TFP and markup shocks, I first provide an analysis of responses to the monetary policy shock under the simple policy rule. I compare model responses to the evidence from empirical studies that document effects of exogenous changes in interest rates on households. Using an exogenous monetary policy shock is a natural way to see the differential effect on the households' income, saving, and consumption. Insights from this exercise help understand economics behind the Ramsey policy later.

I look at the response of the model to 1% contractionary monetary policy shock (Figure 3). I use the simple policy rule $i_t = r_t + 1.5\pi_t + \varepsilon_t$ to close the model, where ε_t is a persistent monetary policy shock[18]. Under this simple rule, in Figure 4a, I plot the change of households' income shares in response to the contractionary monetary policy shock.

Contractionary monetary policy increases real interest rates. As a result of that, the financial income share decreases for borrowers and increases for lenders. At the same time, changes in the rest of income components has the opposite effect. This model implication qualitatively aligns with the empirical evidence from Holm et al. (2021).

Moreover, when looking at changes in disposable income, consumption, and investment choices in Figure 4b, the model's predictions can be compared to the data as well. Disposable income decreases for all households, but more so for the poor. In the model, the reduction of the consumption share is relatively the same, given different bond holdings. Disposable income

[18] For the values of the persistence of all the shocks I use the values from Galí (2015), adjusted for the continuous time formulation.

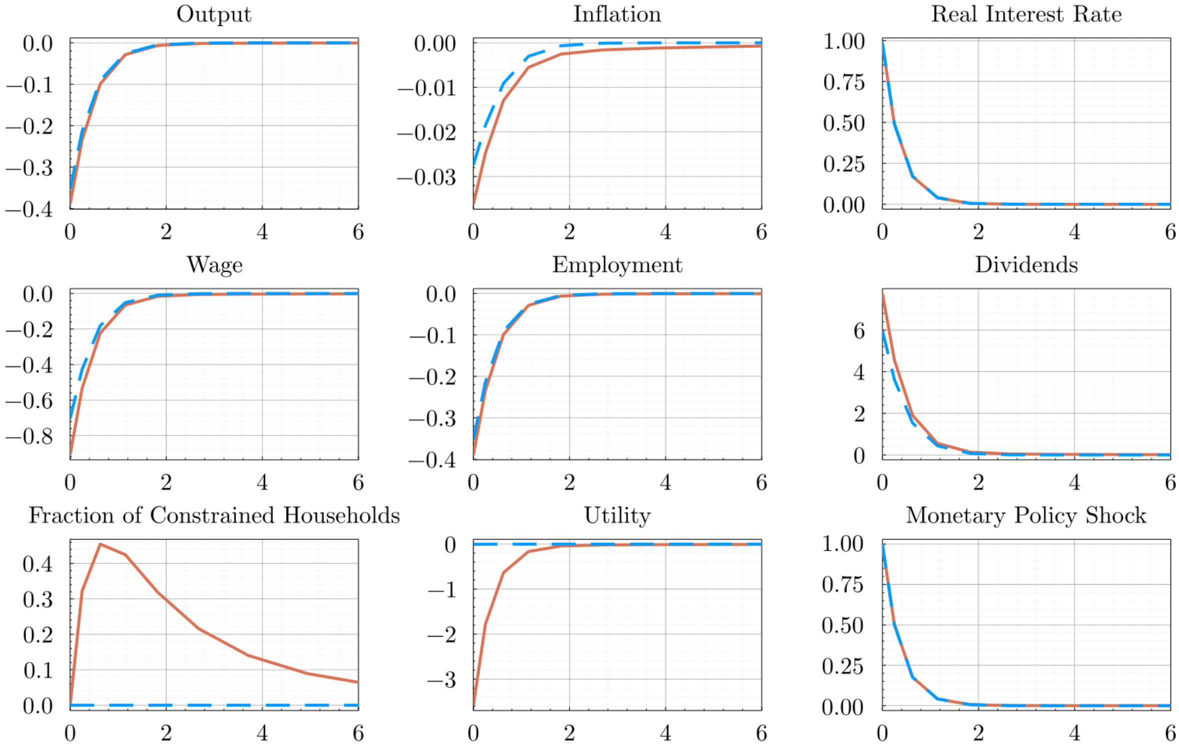


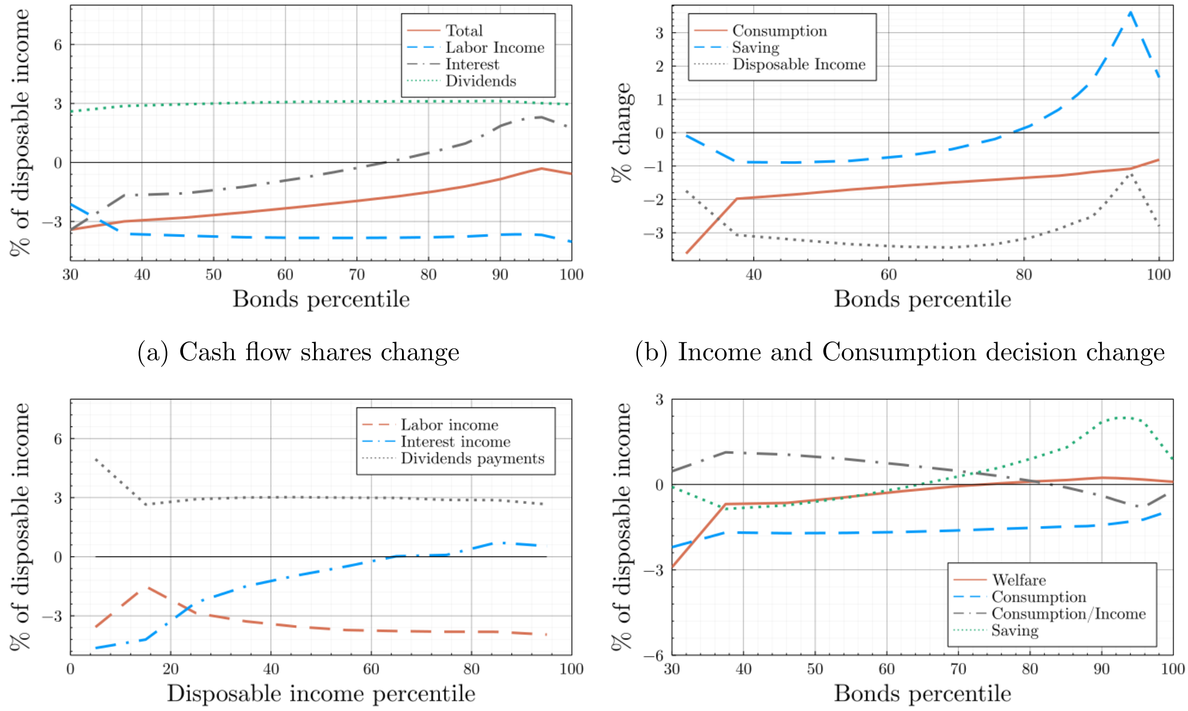
Figure 3: Comparison between responses to the Monetary Policy shock under the simple rule monetary policy in HANK (orange solid line), and in RANK (blue dashed line). Plots show deviations from the steady state values: shock, inflation and real interest rate in percentage points, other variables in percent.

change is transmitted to savings decisions and does not affect the consumption as much as documented empirically. However, crucially, poorer households suffer more from this shock, which is true both in the model and in the data.

Comparison with an empirical evidence from Andersen et al. (2020) cannot deliver the same results. This is because there are no wealthy hand-to-mouth agents in the model. Empirically, a high proportion of households with a high disposable income and a high debt have an increase in debt payments following an increase in real interest rates, which is documented empirically. Since there is only one asset in my model, wealthy individuals have an increase in income from increased returns on bonds, following an increase of real rates. This dimension is at odds with the data because of the simplifying assumption of having one asset in the economy. At the same time, the labor income response is captured quite precisely. It matches a more significant reduction of the labor income for lower income groups after a contractionary monetary policy shock (see Figure 4c).

Finally, comparing model responses with an empirical evidence from Flodén et al. (2020) reveals that households with a higher exposure to interest rate shocks and a higher debt have a more severe decline in consumption (see Figure4d).

Effects of the monetary policy intervention indicate that the model predicts a more detrimental effect on the borrowers than on the lenders. This means that for a given shock that



(a) Cash flow shares change

(b) Income and Consumption decision change

(c) Cash flow shares change

(d) Consumption change

Figure 4: Impact responses to the monetary policy shock

increases the inequality, the monetary authority should optimally implement a looser policy than the strict inflation targeting (SIT), which is optimal in the representative agent NK model. To find a more precise answer and to judge if this incentive is significant enough to alter the optimal policy prescribed by the RANK model, in what follows, I analyze the optimal Ramsey stabilization policy that was calculated numerically using the procedure described in previous sections.

5.3 Optimal policy

I calculate the Ramsey policy under the full commitment in response to an unanticipated MIT shock. The Ramsey planner is restricted by keeping past commitments from before an arrival of a shock. I initialize these commitments by their steady state values. This means that the optimal policy is the optimal stabilization policy. The target for such a policy is to minimize deviations from the steady state in terms of aggregate variables and individual outcomes. This implies that the planner will aim to reduce the inequality to initial levels for a shock that increases the inequality, but the opposite also holds. Since this inequality can be affected by an unexpected monetary policy intervention, the time inconsistency arises. This method allows to find the optimal response to the shocks alone, abstracting from the redistribution considerations in the steady state.

An alternative method that is used in models with the time inconsistency is an introduction of the redistribution scheme that eliminates the time inconsistency. It allows the analysis of an unconstrained Ramsey policy in response to shocks. This method has two crucial shortcomings for the implication in the model with the binding borrowing constraints. First, the method that I am using gives an advantage of studying implications for the model that has a positive inequality in the steady state and allows the monetary policy to influence the inequality level. Second, since the main feature of this model is the endogenous distribution of households, the full insurance provided by such redistribution policy will result in a degenerate steady state distribution of bonds, which eliminates the main feature of the model.

For shocks that reduce inequality, the optimal policy will aim to maintain an initial level of inequality, hence taking the action to increase it after an impact of a shock. Taking this into account, for both TFP and markup shocks, I choose the contractionary shock so that the overall effect increases inequality under the SIT policy and allows me to illustrate the mechanisms at play for the reduction of inequality that the Ramsey planner chooses.[19] At the same time, it is true that for positive shocks, the optimal stabilization policy will imply a symmetrical and opposite response.[20]

Negative TFP shock

I first analyze optimal responses to the TFP shock (see the markup shock analysis in the Appendix) and perform the comparison with the RANK model. I take the shock that reduces the productivity by 1% on impact and reverts back to the steady state level at a rate $e^{-0.42t}$, which is analogous to the process assumed in Galí (2015) but in continuous time, instead of the quarterly discretization. Specifically, as seen in Figure 5 in response to a negative TFP shock, the optimal RANK model policy implies the zero inflation or the strict inflation targeting (SIT) policy. Optimal policy in HANK model instructs to respond to the shock with a positive inflation and then reduce the inflation to negative values before returning to the steady state level. Notice that even though the inflation has been negative for some time, the new nominal price level will differ from the initial steady state level after the shock has passed.

The reason behind this response can potentially have two explanations: either the SIT policy in HANK fails to achieve the same behavior of aggregates, and a different policy is needed to sustain the same optimal response, or the notion of optimal response is different. By comparing paths of aggregate variables in HANK model, where the monetary authority follows the SIT, I conclude that the latter is the case. The different target for the optimal response can be explained by a new redistributive motive, which calls for a change in the SIT policy that achieves the zero output gap. When comparing responses of RANK and HANK models

[19] In general, it is not guaranteed that the contractionary shock increases inequality in a given model with heterogeneity. This question was carefully studied by Bilbiie (2019) in a model with two households and exogenous transition probability between the constrained and unconstrained states. That paper demonstrates that different compositions of income, as well as the structure of taxes, can result in pro or countercyclical inequality. The heterogeneous agent model delivers good predictions under countercyclical inequality, which is a feature of my model.

[20] The response is symmetrical for small enough deviations from the steady state. Since the solution method is fully non-linear, it allows for identifying differences in the optimal policy between big contractionary and expansionary shocks. This study is left outside of the scope of this paper.

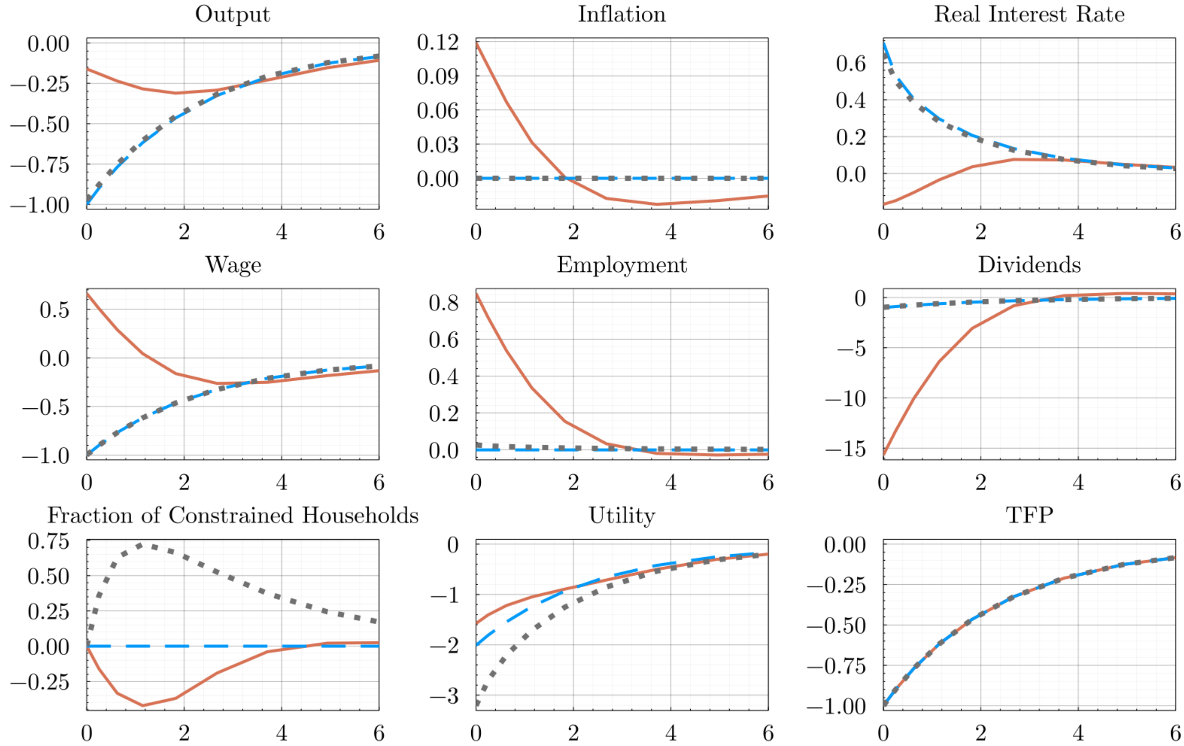


Figure 5: Comparison between responses to the TFP shock under the optimal monetary policy (OP) in HANK (orange solid line), OP in RANK (blue dashed line), and OP from RANK implemented in HANK (black dotted line). Plots show deviations from the steady state values: shock, inflation and real interest rate in percentage points, other variables in percent.

with SIT to the same TFP shock, it is evident from the response of the aggregate variables that the effect of the same SIT policy is almost identical in two models. Since the optimal policy in HANK is substantially different from SIT and calls for a non-zero response in inflation, it implies that having heterogeneous agents in the economy introduces a new source of inefficiency that has to be addressed by the optimal monetary policy, additional to the usual output gap and price stabilization.

Specifically, when looking at the result of the policy, the crucial difference between RANK and HANK optimal responses is in the real interest rate responses. In the HANK model the real interest rate decreases, when in the RANK model it increases. This creates a redistribution towards low wealth individuals and, more specifically, towards constrained agents. The result of such redistribution can be seen in the fact that under the optimal policy in HANK, the fraction of constrained agents is falling as opposed to implementing the strict inflation targeting in HANK. Notice that higher wages and lower dividends and transfers under the optimal policy in HANK also imply redistribution. Here the redistributive effect of a lower interest rate is combined with the redistributive force of wages and dividends. In the model where the distribution of dividends is uncorrelated with wealth, this redistributive force will be lower or even reversed. However, the optimal policy is still calling for an expansionary monetary policy.[21]

[21]Despite the lower nominal rate after the shock, the ZLB is not hit. Impulse response functions of other variables can be found in Appendix.

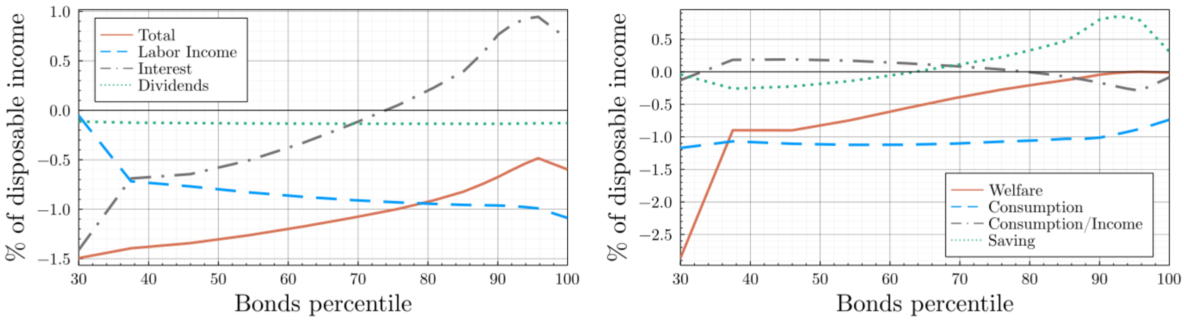


Figure 6: Disposable income and consumption-saving policy response under SIT

To have a better understanding of the redistributional effect of the optimal monetary policy, I look at the heterogeneous response of household disposable income and consumption-saving decisions after a shock hits the economy. In Figure 6, the left graph shows the baseline income redistribution in the HANK model under SIT, immediately after the impact of the TFP shock. Comparing disposable income components with their steady state values reveals that an increase in the real interest rate creates the redistribution from poor to rich, as seen from the change in interest payments. Reduction of wages creates the redistribution in the opposite direction, as seen from the labor income change. Importantly, the redistribution created by the real interest rate dominates, and the total negative effect on disposable income is unevenly distributed among the bond households, hitting mostly the poor.

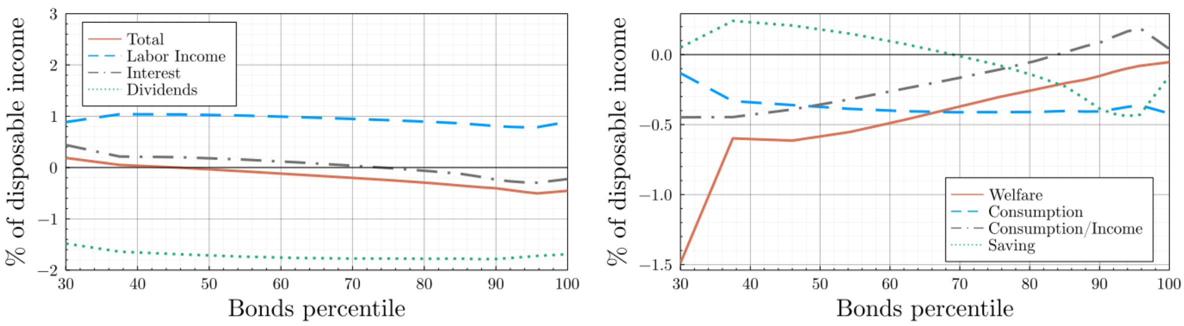


Figure 7: Disposable income and consumption-saving policy response under Ramsey

The graph on the right shows the effect such redistribution has on the consumption-saving decision of households as well as their respective welfare. A more severe reduction in the disposable income leads to a bigger decline in the welfare for poor households, with a significant drop for constrained agents. This drop is caused by their inability to smooth consumption by borrowing, thus severely cutting on consumption and increasing the labor supply, compared to unconstrained agents.

Now, having a baseline in the form of SIT policy, I compare it to the redistribution achieved by the optimal policy. Specifically, Figure 7 shows the redistribution under the Ramsey policy, and Figure 8 compares the disposable income and the consumption-saving decision of households immediately after the TFP shock under the optimal policy with outcomes under SIT.

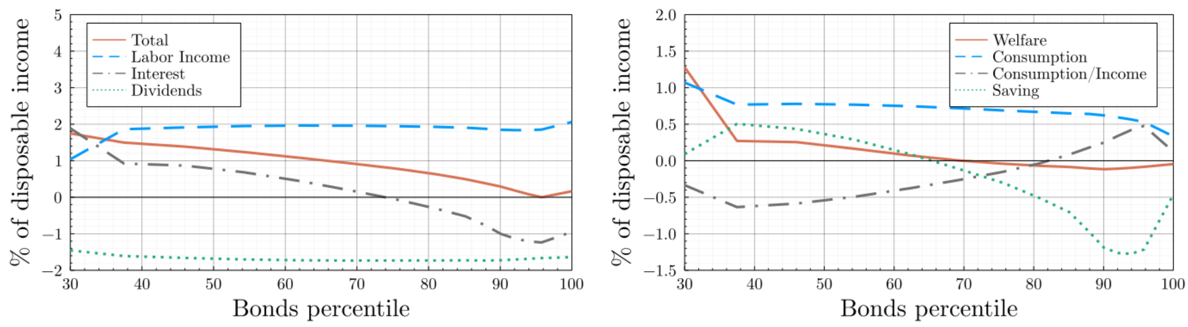


Figure 8: Disposable income and consumption-saving differential response (Ramsey - SIT)

Outcomes immediately after the negative TFP shock under the Ramsey policy compared to the steady state reveals a different redistribution of income that favors the poor. The welfare of households still decreases, but the shock is more evenly loaded on households, reducing the burden on the poorest.

The graph on the left shows that the aggregate disposable income is redistributed more towards households with higher debt which is achieved by implementing a lower real interest rate. On the right-hand side, I show how changes in the disposable income translates to changes in the consumption-saving decision. In line with the logic discussed above when looking at pure monetary policy shocks, the lower real interest rate under the Ramsey policy creates redistribution towards poor individuals, which translates into higher consumption and higher investment.

5.4 Reduced heterogeneity

I compare the size of the redistributive effect that calls for a looser monetary policy to two models. First, where the fraction of the constrained households is fixed, as in the two-agent NK (TANK) models. Second, where it is zero, as in Bhandari et al. (2021): a HANK model with the natural borrowing limit (NB HANK). Specifically, in the adequately calibrated TANK model, with both agents having different labor productivities and different bond holdings[22] (positive for unconstrained and negative for constrained), the effect on income through opposite exposure to interest rates (Figure 9) can be achieved as well.

The borrower pays interest on debt, and the lender receives the return. The crucial difference between HANK and TANK model is the absence of the incentive for Ramsey planner to affect the bond holdings of individuals in the latter. Optimal monetary policy is looser than in the RANK model to achieve some consumption smoothing but less so than in HANK. Quantitatively, taking the difference in inflation response between TANK and HANK models, the reduction of the fraction of constrained agents accounts for 30% of the rise of inflation, or 25% of the difference in the aggregate output response.

[22]This is not a standard assumption for TANK models, but to have a direct comparison with the HANK model, the income effect that comes from the real interest rate exposure has to be considered in the TANK model as well.

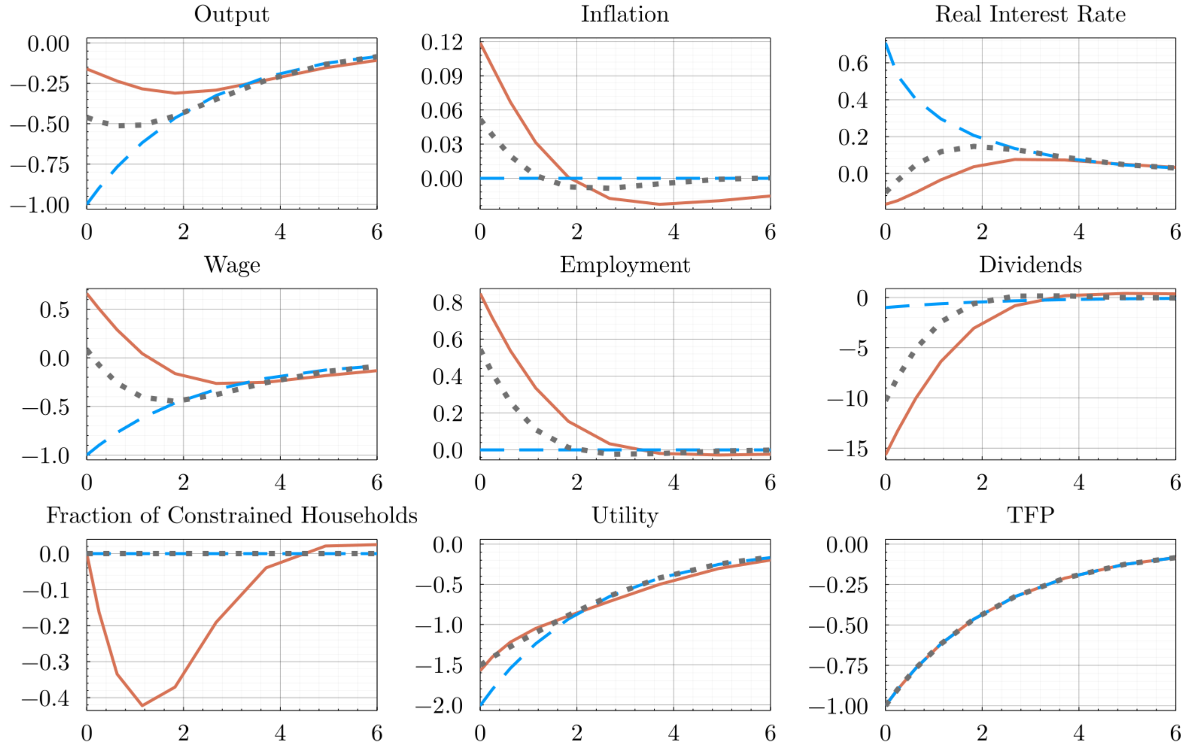


Figure 9: Comparison between responses to the TFP shock under the optimal monetary policy in HANK (orange solid line), RANK (blue dashed line), and TANK (black dotted line). Plots show deviations from the steady state values: shock, inflation and real interest rate in percentage points, other variables in percent.

When comparing HANK model with NB HANK, the difference is less pronounced (Figure 10). Because the calibration in NB HANK implies a higher variance in consumption, it increases the inequality, fueling the incentive to redistribute from dividends to wages. At the same time, the real interest rate still does not fall as much as in the HANK economy under the optimal policy.

The difference that the binding borrowing constraint brings into the model without it is small for this model calibration. Nevertheless, this holds only for the TFP shock. As shown below, the optimal response to the Markup shock is drastically different between the two models. Moreover, the analysis of the importance of the binding borrowing constraint reveals that if the fraction of constrained households is higher than 35%, even after the TFP shock, the model with the binding borrowing constraint instructs for an even looser monetary policy. More on this later.

Comparison of these three cases illustrates that having a fixed share of constrained agents or heterogeneous agents with a natural borrowing limit does not provide the same incentives for the Ramsey planner to reduce the real interest rate as the HANK model with the occasionally binding borrowing constraint. The reason for this is that in the case of the TANK model, the reduction of real interest rates does reduce the pressure on constrained households, but they cannot react in the same way by starting to save and switching to unconstrained.

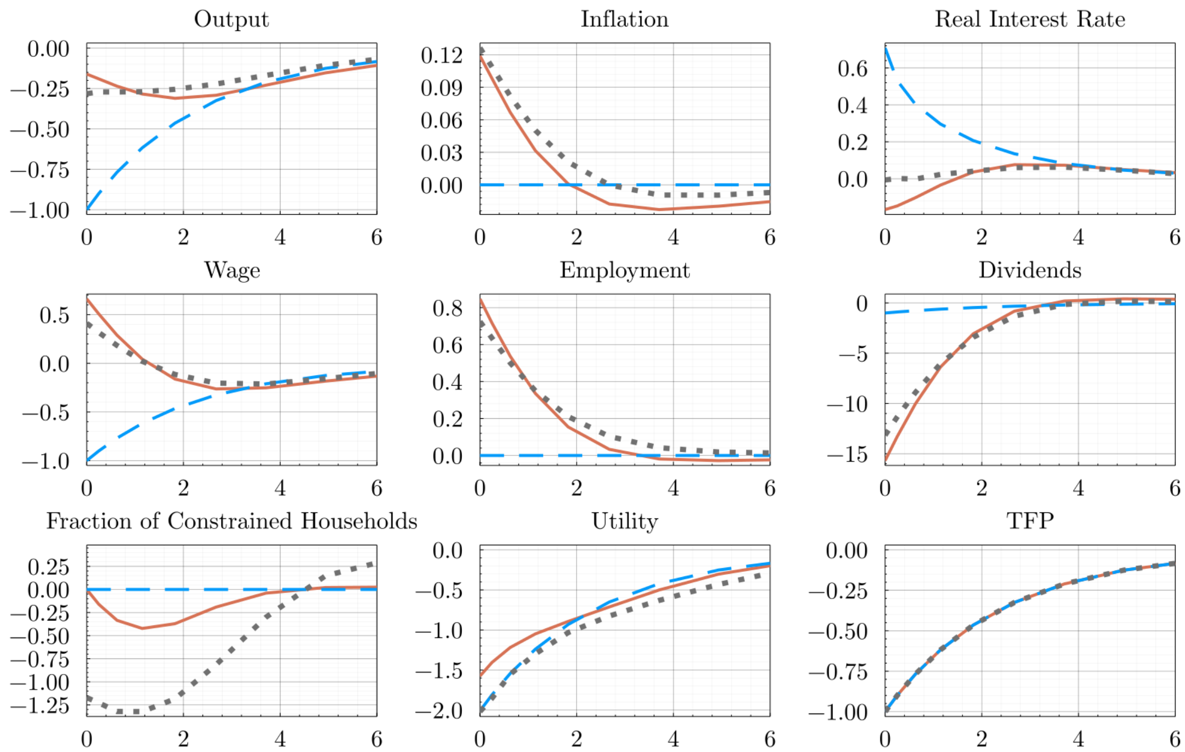


Figure 10: Comparison between responses to the TFP shock under the optimal monetary policy in HANK (orange solid line); RANK (blue dashed line), and NB HANK (black dotted line). Plots show deviations from the steady state values: shock, inflation and real interest rate in percentage points, other variables in percent.

5.5 Importance of the fraction of constrained households

Previous sections indicate that the behavior of the optimal monetary policy is substantially different depending on the fraction of constrained households in the economy. To study this dependence more carefully, I perform the following exercise. I calibrate an array of HANK economies with the differential fraction of constrained households, starting from a few percent, which in the limiting case corresponds to the NB HANK model, to as high as 50%. For these various economies, compute the optimal response to a contractionary TFP shock and plot the deviations of the real interest rate and output on impact, compared to the steady state. The results of this exercise can be seen in Figure 11.

Figures show that $30-35\%$ is a transition range for this economy, after which the Ramsey planner in HANK economy starts to implement a significantly different monetary policy from the Ramsey planner in NB HANK. These graphs illustrate that despite the seemingly small differences between the two calibrated models in the case of a TFP shock, the actual difference can be much more significant if the fraction of constrained households is underestimated.

Notice that this picture contrasts the importance of the occasionally binding borrowing constraint in my model with the exogenous fraction of constrained households in Bhandari et al. (2021). In their paper the optimal policy is almost unaffected by including the fixed share of constrained households, which contrasts with the figures above.

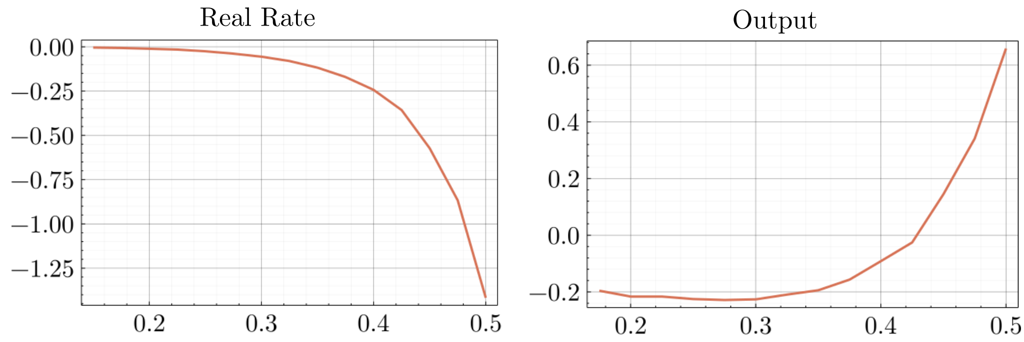


Figure 11: Impact change with respect to the steady state level after the optimal monetary policy response to the TFP shock, depending on the fraction of constrained households in the economy.

6 Conclusion

In this paper I study the optimal monetary policy in HANK with an occasionally binding borrowing constraint, when the fraction of credit constrained households is endogenous. The solution to this problem makes a two-fold contribution to the literature. First, I extend methods used in continuous-time optimal policy literature to an application with a non-trivial binding constraint. Second, I provide a comparison of the implications of this model with the two main alternatives in the literature: TANK model and a HANK model with the natural borrowing limit.

This paper bridges the gap between theoretical and numerical literature that focus on heterogeneity and optimal monetary policy. This bridge is similar in style to Debortoli and Galí (2017), where authors argue that the presence of credit constrained agents has a crucial implication for the positive implications of the monetary policy and argue that TANK economy is good enough to grasp most of it. In this paper I present the analysis from the normative perspective, focusing on the implications for the optimal monetary policy. I show that considering HANK model with an endogenous fraction of constrained individuals significantly changes the optimal Ramsey policy with respect to RANK model. But in case of the optimal policy implications, TANK model does not offer such a good approximation any more. Moreover, a full heterogeneity is not enough either, if the binding borrowing constraint is missing. I show this by performing the comparison with a model that features the natural borrowing limit, as in Bhandari et al. (2021).

I conclude that the presence of an endogenous fraction of constrained households in the model introduces an important margin for the optimal monetary policy to consider. Specifically, the Ramsey planner optimally decides to relax the constraint for as many households as possible in response to contractionary shocks in order to allow them to smooth consumption intertemporaly. As illustrated by comparison of the results with alternative model specifications that do not share this channel, the additional motive for lower interest rate response is substantial. When looking at the model that is calibrated to US data, this channel proves to be one of the most important channels that shapes the optimal monetary policy. Quantitatively, it ranges from 10% to 25% of the overall difference in real interest rate response with respect to the representative agent economy.

References

Acharya, S., E. Challe, and K. Dogra (2021). Optimal Monetary Policy According to HANK. Staff working papers, Bank of Canada.

Acharya, S. and K. Dogra (2020). Understanding HANK: Insights From a PRANK. *Econometrica 88*(3), 1113–1158.

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2021). Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach. *The Review of Economic Studies 89*(1), 45–86.

Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics 109*(3), 659–684.

Amberg, N., T. Jansson, M. Klein, and A. R. Picco (2022). Five Facts about the Distributional Income Effects of Monetary Policy Shocks. *American Economic Review: Insights 4*(3), 289–304.

Andersen, A. L., N. Johannesen, M. Jørgensen, and J.-L. Peydró (2020). Monetary policy and inequality. Economics Working Papers 1761, Department of Economics and Business, Universitat Pompeu Fabra.

Bewley, T. (1983). A Difficulty with the Optimum Quantity of Money. *Econometrica 51*(5), 1485–504.

Bhandari, A., D. Evans, M. Golosov, and T. J. Sargent (2021). Inequality, Business Cycles, and Monetary-Fiscal Policy. *Econometrica 89*(6), 2559–2599.

Bilbiie, F. (2008). Limited asset markets participation, monetary policy and (inverted) aggregate demand logic. *Journal of Economic Theory 140*(1), 162–196.

Bilbiie, F. (2019). Monetary Policy and Heterogeneity: An Analytical Framework. 2019 Meeting Papers 178, Society for Economic Dynamics.

Bilbiie, F. (2020). The New Keynesian cross. *Journal of Monetary Economics 114*(C), 90–108.

Bilbiie, F. and X. Ragot (2021). Optimal Monetary Policy and Liquidity with Heterogeneous Households. *Review of Economic Dynamics 41*, 71–95.

Bilbiie, F. O., T. Monacelli, and R. Perotti (2020). Stabilization vs. Redistribution: the Optimal Monetary-Fiscal Mix. Discussion paper series, CEPR.

Carroll, C. D. (2006). The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems. *Economics Letters 91*(3), 312–320.

Challe, E. (2020). Uninsured Unemployment Risk and Optimal Monetary Policy in a Zero-Liquidity Economy. *American Economic Journal: Macroeconomics 12*(2), 241–83.

Clarida, R., J. Galí, and M. Gertler (1999). The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature 37*(4), 1661–1707.

Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*, Volume 264. Springer.

Dávila, E. and A. Schaab (2022). Optimal Monetary Policy with Heterogeneous Agents: A Timeless Ramsey Approach. Technical report, SSRN.

Debortoli, D. and J. Galí (2017). Monetary policy with heterogeneous agents: Insights from TANK models. Economics Working Papers 1686, Department of Economics and Business, Universitat Pompeu Fabra.

Dixit, A. K. and R. S. Pindyck (1994). *Investment under Uncertainty.* Number 5474 in Economics Books. Princeton University Press.

Flodén, M., M. Kilström, J. Sigurdsson, and R. Vestman (2020). Household Debt and Monetary Policy: Revealing the Cash-Flow Channel. *The Economic Journal 131*(636), 1742–1771.

Galí, J. (2015). *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications Second edition.* Princeton University Press.

Galí, J., J. D. López-Salido, and J. Vallés (2004). Rule-of-Thumb Consumers and the Design of Interest Rate Rules. *Journal of Money, Credit and Banking 36*(4), 739–763.

González, B., G. Nuño, D. Thaler, and S. Albrizio (2022). Firm Heterogeneity, Capital Misallocation, and Optimal Monetary Policy. Working paper no. 2145, Banco de Espana.

Holm, M. B., P. Paul, and A. Tischbirek (2021). The Transmission of Monetary Policy under the Microscope. *Journal of Political Economy 129*(10), 2861–2904.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control 17*(5), 953–969.

Kaplan, G., B. Moll, and G. L. Violante (2018). Monetary Policy According to HANK. *American Economic Review 108*(3), 697–743.

Kaplan, G. and G. L. Violante (2014). A Model of the Consumption Response to Fiscal Stimulus Payments. *Econometrica 82*(4), 1199–1239.

Le Grand, F., A. Martin-Baillon, and X. Ragot (2021). Should monetary policy care about redistribution? Optimal fiscal and monetary policy with heterogeneous agents. Working paper.

Le Grand, F. and X. Ragot (2022). Managing Inequality over the Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks. *International Economic Review 63*(1), 511–540.

Malliavin, P. and A. Thalmaier (2006). *Stochastic Calculus of Variations in Mathematical Finance*. Springer.

Mankiw, N. G. (2000). The Savers-Spenders Theory of Fiscal Policy. *American Economic Review 90*(2), 120–125.

McKay, A. and C. K. Wolf (2022). Optimal Policy Rules in HANK. Working paper.

Nisticó, S. (2016). Optimal Monetary Policy and Financial Stability in a Non-Ricardian Economy. *Journal of the European Economic Association 14*(5), 1225–1252.

Nuño, G. (2017). Optimal Social Policies in Mean Field Games. *Applied Mathematics & Optimization 76*, 29–57.

Nuño, G. and C. Thomas (2022). Optimal Redistributive Inflation. *Annals of Economics and Statistics 146*(146), 3–64.

Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

7 Appendix

7.1 Steady state plots

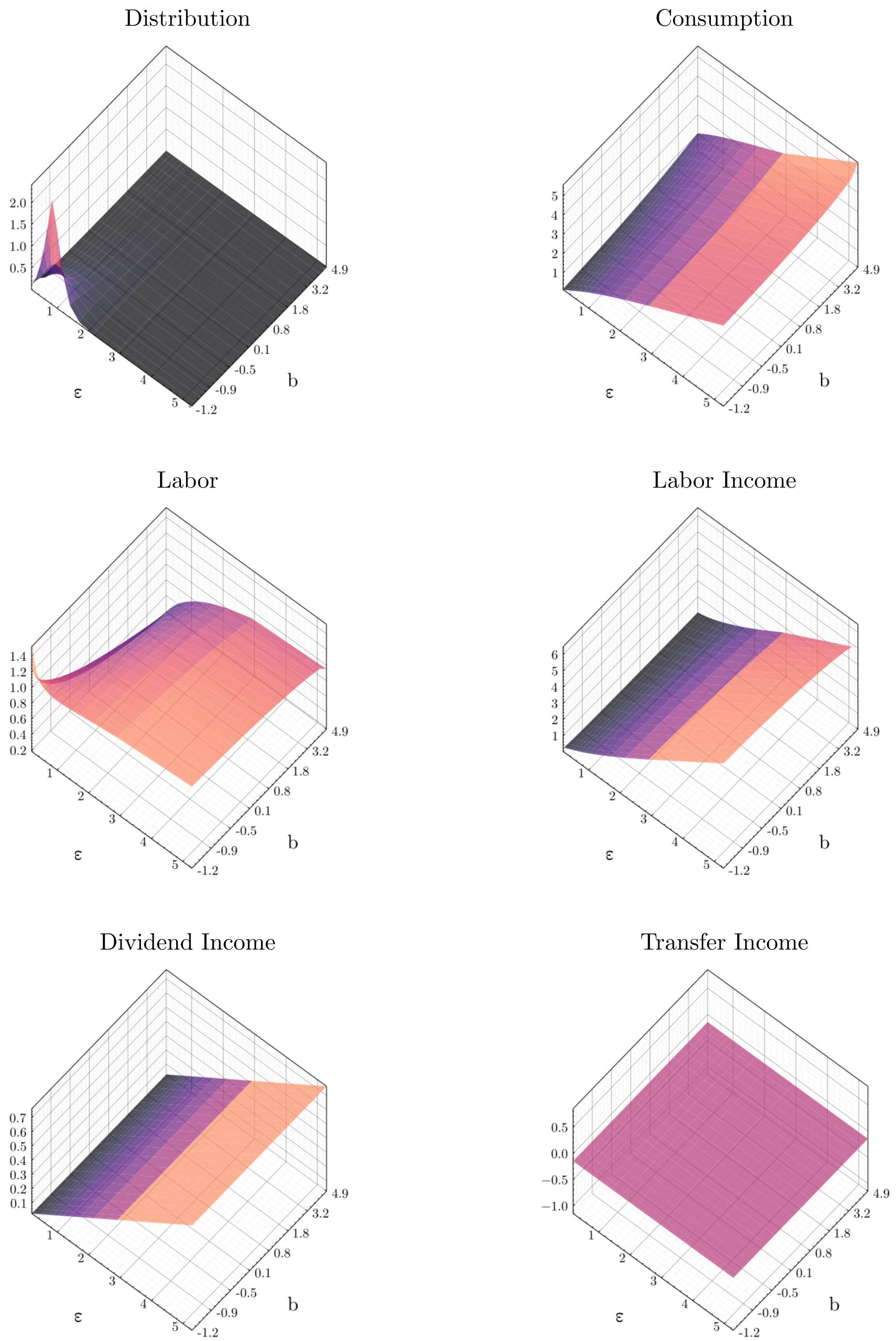


Figure 12: Steady state

7.2 Dynamic response to the inequality-increasing Markup shock under the optimal policy

I study the response to a shock that temporarily increases the price elasticity of demand for intermediate goods by 1, thus reducing desired markups by approximately 1%. The decay of the shock follows $e^{-0.89t}$, which is analogous to the process assumed in Galí (2015). The same logic for creating the redistribution by reducing real interest rates, reducing dividends, and increasing wages, as in the case of the TFP shock, also holds for the markup shock.

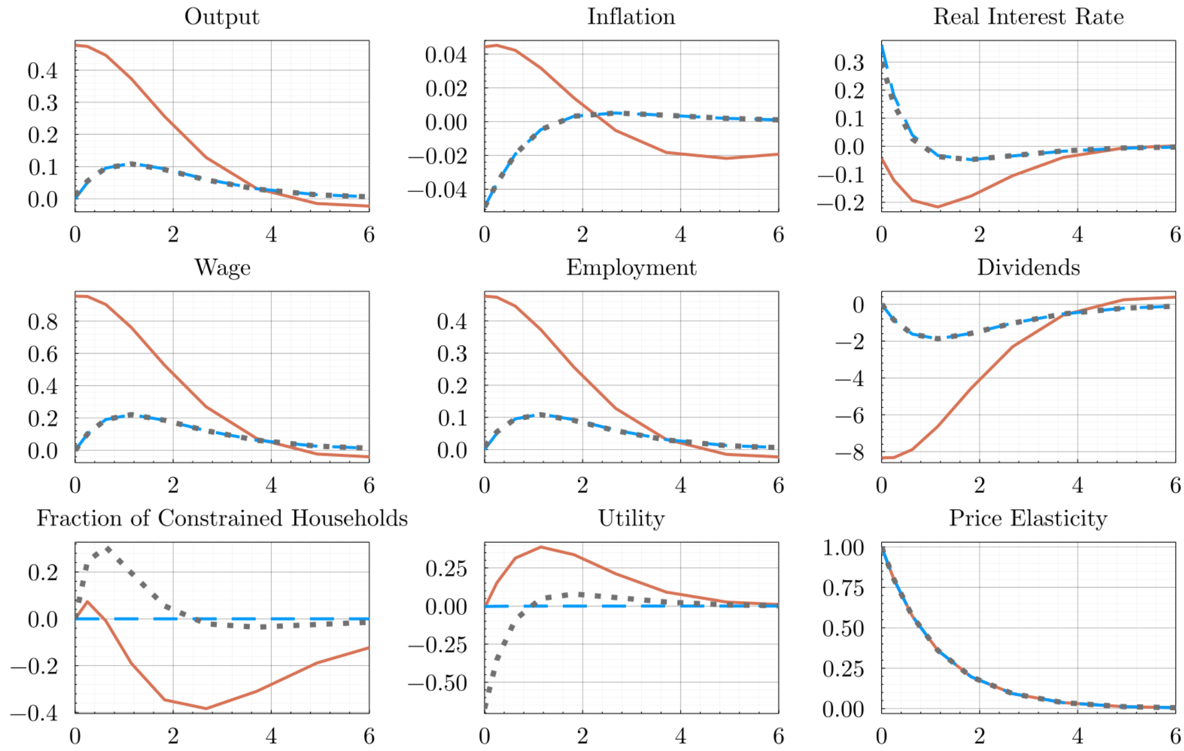


Figure 13: Comparison between responses to the Markup shock under the optimal monetary policy (OP) in HANK (orange solid line), OP in RANK (blue dashed line), and OP from RANK implemented in HANK (black dotted line).

In Figure 13, I compare the responses to markup shock in HANK and RANK under the optimal monetary policy. Importantly, when the optimal monetary policy from RANK is implemented in HANK,[23] the behavior of aggregate variables is the same as in the calibrated RANK model, which, again, confirms the importance of the additional redistributive incentive that appears in the HANK model. Following the same logic of reducing the impact of the inequality-increasing shock, the optimal monetary policy in HANK on impact implements lower real interest rates by 0.3 pp, reduces the dividends payments by 4.5%, and increases real wages by 0.5%. All these effects combined lead to a higher aggregate utility path in HANK than the policy from RANK will otherwise imply. The aggregate welfare improvement is 0.07% over the

[23] In the case of markup shock to implement the policy from RANK in HANK, I assume that the monetary authority matches the path of the real interest rate. Alternative assumptions can imply matching of other aggregate variables, such as inflation or output. However, since the paths of aggregate variables are matched very precisely under this specific rule, any other matching rule will closely reproduce the same outcome.

”lean against the wind” policy.

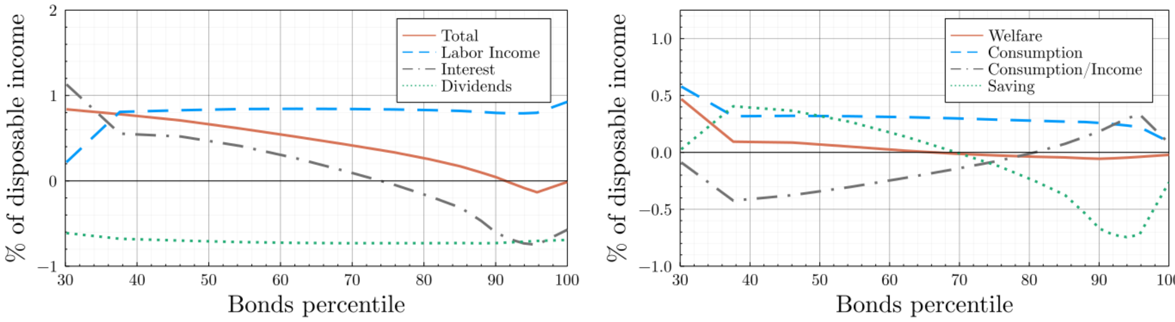


Figure 14: Disposable income and consumption-saving differential response (Ramsey HANK - Ramsey RANK)

In this case, the redistributional motive works by managing the impact on the poor households and relaxing the binding constraint for some households on it. The mechanism is similar to the response to the TFP shock. When comparing the difference in immediate responses after a shock under the optimal policy in HANK to the optimal policy from RANK implemented in HANK in Figure 14 the mechanism of the optimal policy is, again, to create a relative redistribution of the disposable income towards poorer households through lowering real interest rates, which leads to a higher consumption. Between the changes in real interest rates, dividends

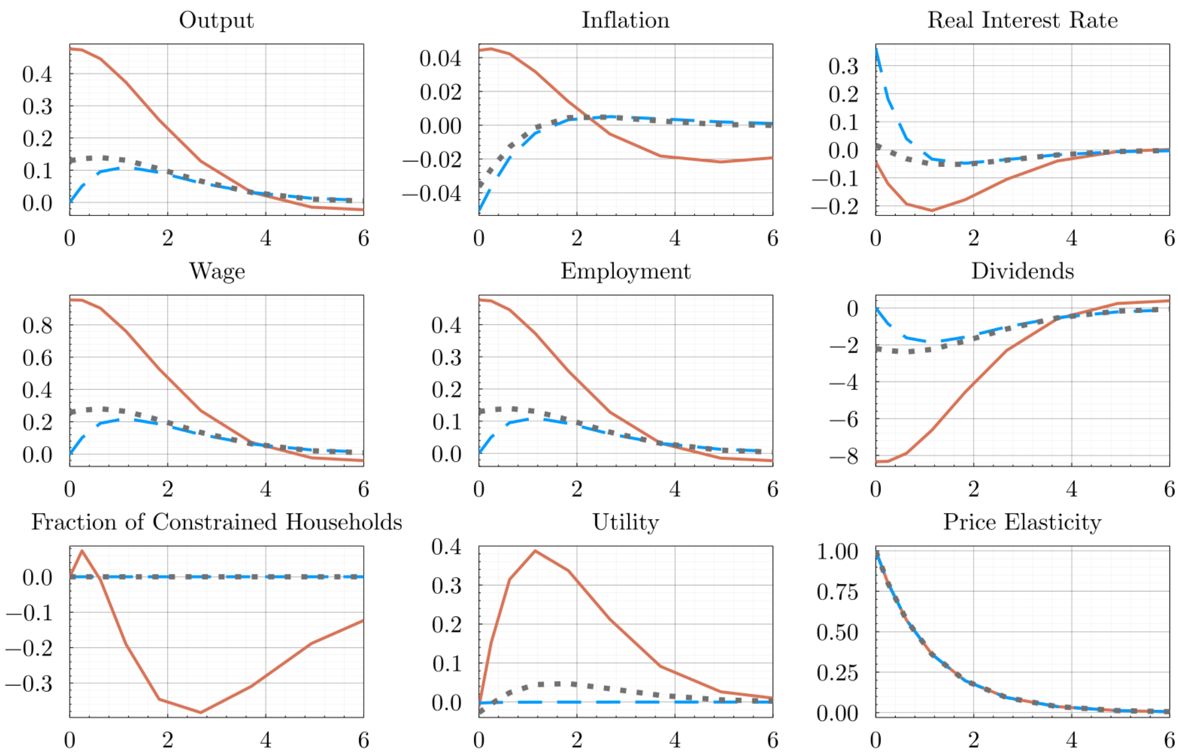


Figure 15: Comparison between responses to the Markup shock under the optimal monetary policy in HANK (orange solid line), RANK (blue dashed line), and TANK (black dotted line).

and wages, the main redistributive effect is achieved by imposing lower real interest rates. The aggregate redistribution effect implemented by the optimal policy in HANK leads to an almost

1% increase in the disposable income for constrained households compared to the policy taken from RANK. Such the redistribution results in an increase in welfare for constrained households by 0.5%.

Comparing results with the calibrated TANK model in Figure 15 shows that similarly to the case of the optimal response to the TFP shock, the TANK model recovers some of dynamics of the optimal policy in HANK. The difference again is attributed to a missing incentive of reduction in the constrained fraction in TANK. This time an additional incentive for a looser monetary policy is more significant. On impact, the channel accounts for 75% of the difference in inflation response and 50% in the difference in the output response. Alternatively, the TANK model can recover at most 30% of the total effect compared to the HANK model. The paths of dividends and wages are closer to HANK, but only on impact: 1 year after the impact of a markup shock, paths are almost identical to the RANK model. This difference highlights the importance of the new incentive of reduction in the fraction of constrained households in the case of a shock to markup. Under the optimal monetary policy in HANK, the fraction of constrained agents reduces by 0.3% with respect to the steady state and by 0.2% with respect to the policy from TANK. When measured in terms of the aggregate welfare, the improvement is 0.05%.

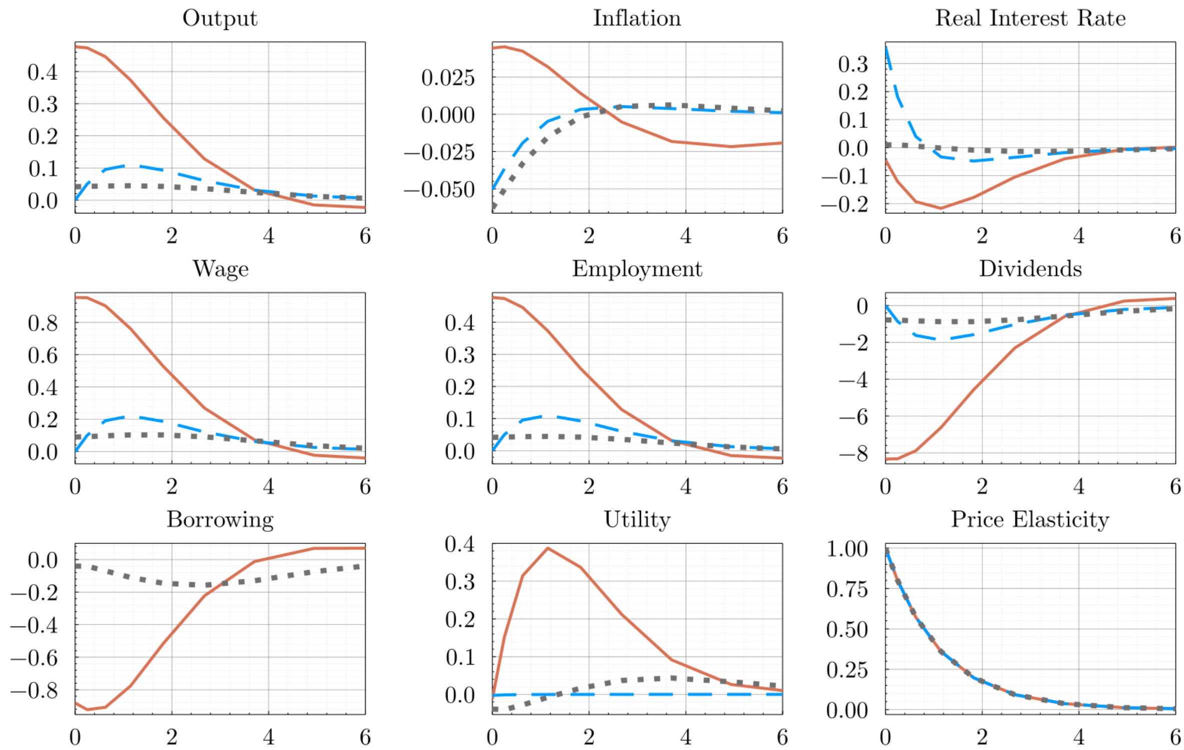


Figure 16: Comparison between responses to the Markup shock under the optimal monetary policy (OP) in HANK (orange solid line), RANK (blue dashed line), and NB HANK (black dotted line).

Finally, comparing this result to the optimal policy in HANK with NB HANK in Figure 16 shows that in the economy with a natural borrowing limit, the optimal monetary policy can almost completely counteract the effect of the markup shock. Real interest rates, dividends, and wages almost do not change relative to the steady state. The incentive for reducing the real interest rate is strongest in the case of the model with the occasionally binding borrowing constraint. Reducing the fraction of constrained households changes the target for the Ramsey planner and leads to a looser monetary policy. This effect is much stronger than for the TPF shock discussed in the previous section. In terms of wages and dividends, the NB HANK model is further from the HANK prediction than the RANK economy. In contrast, the real interest rate has the same behavior as in the TANK model, not offering any significant improvement over the simpler model.

7.3 Household law of motion adjoins to KF

In the appendix I will drop the time and household subscripts for the values. Though in some cases no subscripts means that the object is a function rather than a value of a function. The exact use will be clear each time and this simplification is done in order to shorten the expressions.

For the two given functions g, h the proof is the following

$$\langle g, \mathcal{A}^* h\rangle = \int\limits_{\varepsilon}\int\limits_{b} g\mathcal{A}^* h db d\varepsilon = \tag{36}$$

$$= \int\limits_{\varepsilon}\int\limits_{b} g\left(-\frac{\partial}{\partial b}\left\{\left(\lambda W l\varepsilon + d\varepsilon + T + r^b b - c\right)h\right\} - \frac{\partial}{\partial\varepsilon}\rho_\varepsilon\varepsilon(\bar{e} - e)h + \frac{\partial^2}{\partial\varepsilon^2}\varepsilon\frac{\sigma_\varepsilon^2}{2}h\right) db d\varepsilon = \tag{37}$$

$$= -\int\limits_{\varepsilon}\int\limits_{b} g\frac{\partial}{\partial b}\dot{b}h db d\varepsilon - \int\limits_{b}\int\limits_{\varepsilon} g\frac{\partial}{\partial\varepsilon}\dot{\varepsilon}h d\varepsilon db + \int\limits_{b}\int\limits_{\varepsilon} g\frac{\partial^2}{\partial\varepsilon^2}\frac{\sigma_\varepsilon^2}{2}h d\varepsilon db = \tag{38}$$

$$= \int\limits_{\varepsilon}\int\limits_{b}\dot{b}h\frac{\partial}{\partial b}g\ db d\varepsilon - \int\limits_{\varepsilon}\left[g\dot{b}f\Big|_{\underline{b}}^{\infty}\right] d\varepsilon + \int\limits_{b}\int\limits_{\varepsilon}\dot{\varepsilon}h\frac{\partial}{\partial\varepsilon}g\ d\varepsilon db - \int\limits_{b}\left[g\dot{\varepsilon}h|_0^\infty\right] db \tag{39}$$

$$+ \int\limits_{b}\int\limits_{\varepsilon}\frac{\sigma_\varepsilon^2}{2}h\frac{\partial^2}{\partial\varepsilon^2}g\ d\varepsilon db + \int\limits_{b}\left[g\frac{\partial}{\partial\varepsilon}\frac{\sigma_\varepsilon^2}{2}h\Big|_0^\infty\right] db - \int\limits_{b}\left[\frac{\sigma_\varepsilon^2}{2}h\frac{\partial}{\partial\varepsilon}g\Big|_0^\infty\right] db = \tag{40}$$

$$= \langle\mathcal{A}g, h\rangle - \int\limits_{\varepsilon}\left[g\dot{b}h\Big|_{\underline{b}}^{\infty}\right] d\varepsilon - \int\limits_{b}\left[g\dot{\varepsilon}h|_0^\infty\right] db + \int\limits_{b}\left[g\frac{\partial}{\partial\varepsilon}\frac{\sigma_\varepsilon^2}{2}h\Big|_0^\infty\right] db - \int\limits_{b}\left[\frac{\sigma_\varepsilon^2}{2}h\frac{\partial}{\partial\varepsilon}g\Big|_0^\infty\right] db \tag{41}$$

Notice that when one of the functions is a distribution of the households, it must be that it is equal to zero on the limits of the domain. This implies that the only non-zero boundary integral is $\int_\varepsilon \lim g\dot{b}h\Big|_{b=\underline{b}} d\varepsilon$ This derivation together with the constraint on the density function on the limits of the domain is going to be crucially important for the derivation of the optimal policy.

7.4 Ramsey problem in RANK

This section shows how to solve the RANK optimal MP Ramsey problem using Calculus of Variations.

RANK model is a baseline three equation model in continuous time.

The household problem is the same as in the HANK described in the main text, with constant productivity $\varepsilon = 1$.

$$\mathcal{V}_t = \underset{\{c,l,\dot{b}\}_t}{max}\,\mathbb{E}_t\int\limits_0^{+\infty} e^{-\rho t} Z_t\left(\frac{c_t^{1-\nu}}{1-\nu} - \varphi\frac{l_t^{1+\gamma}}{1+\gamma}\right) dt \tag{42}$$

$$\text{s.t.}\quad c_t + \dot{b}_t = \lambda W_t l_t + d_t + T_t + r_t^b b_t \tag{43}$$

Solving this problem for the representative household and zero net supply of bonds, gives the Euler equation and the labor supply equation.

$$\begin{aligned} r_t^b =& \rho + \frac{\dot{Z}_t}{Z_t} - \nu \frac{\dot{c}_t}{c_t} & (44) \\ l_t^\gamma c_t^\nu =& \frac{\lambda W_t}{\varphi} & (45) \end{aligned}$$

Supply side stays the same, so the Phillips Curve is unchanged.

$$\left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1} m_t - 1\right) + \dot{\pi}_t \tag{46}$$

The Ramsey problem has the following form:

$$\mathcal{L}[c, l, W, Y, \pi, r^b] = \tag{47}$$

$$= \int_0^\infty e^{-\rho t}\left[Z_t\left(\frac{c_t^{1-\nu}}{1-\nu} - \varphi\frac{l_t^{1+\gamma}}{1+\gamma}\right) + \varrho_t\left(r_t^b - \rho + \frac{\dot{Z}_t}{Z_t} - \nu\frac{\dot{c}_t}{c_t}\right) + \mu_{i,t}\left(l_t^\gamma c_t^\nu - \frac{\lambda W_t}{\varphi}\right) + \right. \tag{48}$$

$$+ \eta_{Y,t}\left(Y_t - \theta_t l_t\right) + \eta_{T,t}\left(c_t - \left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t\right) + \tag{49}$$

$$\left. + \eta_{\pi,t}\left(\frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1}\frac{W_t}{\theta_t} - 1\right) + \dot{\pi}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t\right)\right] dt \tag{50}$$

7.4.1 Control over consumption

According to Calculus of Variations, I start with taking the first variation wrt the control function at hand. This implies taking the variation of the function and separately the variation of the variation of the derivative of this function. Notice that here I can take all the variations separately, as they do not interact between each other and the strong forms can be derived separately for each of the controls. This will not be the case in the HANK problem.

Weak form:

$$\frac{\delta\mathcal{L}}{\delta c} = \int_0^\infty e^{-\rho t}\left[Z_t c_t^{-\nu} v_{c,t} + \nu\frac{\varrho_t \dot{c}_t}{c_t^2} v_{c,t} - \nu\varrho_t\frac{\dot{v}_{c,t}}{c_t} + \nu\mu_t l_t^\gamma c_t^{\nu-1} v_{c,t} + \eta_{T,t} v_{c,t}\right] dt \tag{51}$$

To proceed further, I have to isolate the all the variations in their initial form without derivatives.

Rearranging the term with $\dot{v}_{c,t}$:

$$\int_0^\infty e^{-\rho t}\left[-\nu\varrho_t\frac{\dot{v}_{c,t}}{c_t}\right] dt = -e^{-\rho t}\nu\varrho_t\frac{v_{c,t}}{c_t}\Big|_0^\infty + \int_0^\infty e^{-\rho t}\left[\nu\frac{\dot{\varrho}_t}{c_t} - \nu\varrho_t\frac{\dot{c}_t}{c_t^2} - \rho\nu\frac{\varrho_t}{c_t}\right] v_{c,t} dt \tag{52}$$

Finally weak form is:

$$\frac{\delta \mathcal{L}}{\delta c} = \int_0^\infty e^{-\rho t} \left[Z_t c_t^{-\nu} + \nu \frac{\dot{\varrho}_t}{c_t} - \rho\nu \frac{\varrho_t}{c_t} + \nu \mu_t l_t^\gamma c_t^{\nu-1} + \eta_{T,t} \right] v_{c,t} dt - e^{-\rho t} \nu \varrho_t \frac{v_{c,t}}{c_t} \Big|_0^\infty = 0 \tag{53}$$

Now using the standard CV approach to allow for any admissible variation, we can conclude that at any point the term that multiplies the variation has to be zero. Thus the strong form:

$$\begin{cases} \rho \varrho_t = \dot{\varrho}_t + Z_t \dfrac{c_t^{1-\nu}}{\nu} + \mu_t l_t^\gamma c_t^\nu + \dfrac{\eta_{T,t} c_t}{\nu} \\ \lim\limits_{t\to\infty} e^{-\rho t} \nu \dfrac{\varrho_t}{c_t} = 0 \\ \varrho_t \Big|_{t=0} = 0 \end{cases} \tag{54}$$

7.4.2 Control over labor

Weak form:

$$\frac{\delta \mathcal{L}}{\delta l} = \int_0^\infty e^{-\rho t} \left[-Z_t \varphi l_t^\gamma v_{l,t} + \gamma \mu_t l_t^{\gamma-1} c_t^\nu v_{l,t} - \eta_{Y,t} \theta_t v_{l,t} \right] dt = 0 \tag{55}$$

Strong form:

$$-Z_t \varphi l_t^\gamma + \gamma \mu_t l_t^{\gamma-1} c_t^\nu - \eta_{Y,t} \theta_t = 0 \tag{56}$$

7.4.3 Control over wage

Weak form:

$$\frac{\delta \mathcal{L}}{\delta W} = \int_0^\infty e^{-\rho t} \left[-\mu_t \frac{\lambda}{\varphi} v_{W,t} + \eta_{\pi,t} \frac{\phi_t}{\psi \theta_t} v_{W,t} \right] dt = 0 \tag{57}$$

Strong form:

$$-\mu_t \frac{\lambda}{\varphi} + \eta_{\pi,t} \frac{\phi_t}{\psi \theta_t} = 0 \tag{58}$$

7.4.4 Control over return

Weak form:

$$\frac{\delta \mathcal{L}}{\delta r^b} = \int_0^\infty e^{-\rho t} \left[\varrho_t v_{r^b,t} - \eta_\pi \pi_t v_{r^b,t} \right] dt = 0 \tag{59}$$

Strong form:

$$\varrho_t = \eta_\pi \pi_t \tag{60}$$

7.4.5 Control over output

Weak form:

$$\frac{\delta \mathcal{L}}{\delta Y} = \int_0^\infty e^{-\rho t} \left[-\eta_{T,t} \left(1 - \frac{\psi}{2} \pi_t^2 \right) v_{Y,t} + \eta_{Y,t} v_{Y,t} + \eta_{\pi,t} \pi_t \frac{\dot{v}_{Y,t}}{Y_t} - \eta_{\pi,t} \pi_t \frac{\dot{Y}_t}{Y_t^2} v_{Y,t} \right] dt \tag{61}$$

In the same way as in the control over the consumption, to proceed further, I have to isolate all the variations in their initial form without derivatives.

Rearranging the term with $\dot{v}_{Y,t}$:

$$\int_0^\infty e^{-\rho t} \left[\eta_{\pi,t} \pi_t \frac{\dot{v}_{Y,t}}{Y_t} \right] dt = e^{-\rho t} \eta_{\pi,t} \pi_t \frac{v_{Y,t}}{Y_t} \Big|_0^\infty - \tag{62}$$

$$- \int_0^\infty e^{-\rho t} \left[\frac{\frac{\partial}{\partial t}(\eta_{\pi,t} \pi_t)}{Y_t} - \frac{\eta_{\pi,t} \pi_t \dot{Y}_t}{Y_t^2} - \rho \frac{\eta_{\pi,t} \pi_t}{Y_t} \right] v_{Y,t} dt \tag{63}$$

This implies the weak form to be:

$$\frac{\delta \mathcal{L}}{\delta Y} = \int_0^\infty e^{-\rho t} \left[-\eta_{T,t} \left(1 - \frac{\psi}{2} \pi_t^2 \right) + \eta_{Y,t} - \frac{\frac{\partial}{\partial t}(\eta_{\pi,t} \pi_t)}{Y_t} + \rho \frac{\eta_{\pi,t} \pi_t}{Y_t} \right] v_{Y,t} dt + \tag{64}$$

$$+ e^{-\rho t} \eta_{\pi,t} \pi_t \frac{v_{Y,t}}{Y_t} \Big|_0^\infty = 0 \tag{65}$$

Strong form:

$$\begin{cases} \rho \eta_{\pi,t} \pi_t = \frac{\partial}{\partial t}(\eta_{\pi,t} \pi_t) + \eta_{T,t} \left(1 - \frac{\psi}{2} \pi_t^2 \right) Y_t - \eta_{Y,t} Y_t \\ \lim\limits_{t \to \infty} e^{-\rho t} \frac{\eta_{\pi,t} \pi_t}{Y_t} = 0 \\ \eta_{\pi,t} \pi_t \Big|_{t=0} = 0 \end{cases} \tag{66}$$

7.4.6 Control over inflation

Weak form:

$$\frac{\delta \mathcal{L}}{\delta \pi} = \int_0^\infty e^{-\rho t} \left[-\eta_{T,t} \psi \pi_t Y_t v_{\pi,t} - \eta_{\pi,t} \left(r_t^b - \frac{\dot{Y}_t}{Y_t} - \rho \right) - \dot{\eta}_{\pi,t} v_{\pi,t} \right] dt + e^{-\rho t} \eta_{\pi,t} v_{\pi,t} \Big|_0^\infty = 0 \tag{67}$$

Strong form:

$$\begin{cases} \dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \frac{\dot{Y}_t}{Y_t} \right) \eta_{\pi,t} + \eta_{T,t} \psi \pi_t Y_t \\ \lim\limits_{t \to \infty} e^{-\rho t} \eta_{\pi,t} = 0 \\ \eta_{\pi,t} \Big|_{t=0} = 0 \end{cases} \tag{68}$$

7.4.7 Optimal MP policy

Finally all conditions can be brought together in one system:

$$\begin{cases} \rho\varrho_t = \dot{\varrho}_t + Z_t\dfrac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \dfrac{\eta_{T,t}c_t}{\nu} \\ \lim\limits_{t\to\infty} e^{-\rho t}\varrho_t = 0 \\ \varrho_t\Big|_{t=0} = 0 \\ -Z_t\varphi l_t^{\gamma} + \gamma\mu_t l_t^{\gamma-1}c_t^{\nu} - \eta_{Y,t}\theta_t = 0 \\ -\mu_t\frac{\lambda}{\varphi} + \eta_{\pi,t}\frac{\phi_t}{\psi\theta_t} = 0 \\ \varrho_t = \eta_\pi\pi_t \\ \rho\eta_{\pi,t}\pi_t = \frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t) + \eta_{T,t}\left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t - \eta_{Y,t}Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\dfrac{\eta_{\pi,t}\pi_t}{Y_t} = 0 \\ \eta_{\pi,t}\pi_t\Big|_{t=0} = 0 \\ \dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{T,t}\psi\pi_t Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\ \eta_{\pi,t}\Big|_{t=0} = 0 \end{cases} \tag{69}$$

Combining differential equation for output control, inflation control and PC gives

$$\eta_{Y,t} = \eta_{T,t}\left(1 + \frac{\psi}{2}\pi_t^2\right) - \frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1}\frac{W_t}{\theta_t} - 1\right)\frac{1}{Y_t}\eta_{\pi,t} \tag{70}$$

And combining consumption control with output control and real return control

$$Z_t\frac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \frac{\eta_{T,t}c_t}{\nu} = \eta_{T,t}\left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t - \eta_{Y,t}Y_t \tag{71}$$

So the Optimal MP problem becomes:

$$\begin{cases} Z_t\dfrac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \dfrac{\eta_{T,t}c_t}{\nu} = \eta_{T,t}\left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t - \eta_{Y,t}Y_t \\ -Z_t\varphi l_t^{\gamma} + \gamma\mu_t l_t^{\gamma-1}c_t^{\nu} - \eta_{Y,t}\theta_t = 0 \\ -\mu_t\dfrac{\lambda}{\varphi} + \eta_{\pi,t}\dfrac{\phi_t}{\psi\theta_t} = 0 \\ \eta_{Y,t} = \eta_{T,t}\left(1 + \dfrac{\psi}{2}\pi_t^2\right) - \dfrac{\phi_t - 1}{\psi}\left(\dfrac{\phi_t}{\phi_t - 1}\dfrac{W_t}{\theta_t} - 1\right)\dfrac{1}{Y_t}\eta_{\pi,t} \\ \dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{T,t}\psi\pi_t Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\ \eta_{\pi,t}\Big|_{t=0} = 0 \end{cases} \tag{72}$$

Rearranging the first three equations:

$$
\begin{cases}
\mu_t = \dfrac{Z_t \varphi l_t}{\gamma c_t^{\nu}} + \eta_{Y,t} \dfrac{\theta_t}{\gamma l_t^{\gamma-1} c_t^{\nu}} \\
\eta_{\pi,t} = \mu_t \dfrac{\lambda \psi \theta_t}{\phi_t \varphi} \\
\eta_{T,t} = \dfrac{Z_t \frac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \eta_{Y,t} Y_t}{\left(1 - \frac{1}{\nu}\right)\left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t} \\
\eta_{Y,t} = \eta_{T,t} \left(1 + \dfrac{\psi}{2}\pi_t^2\right) - \eta_{\pi,t} \dfrac{1}{Y_t} \dfrac{\phi_t - 1}{\psi} \left(\dfrac{\phi_t}{\phi_t - 1} \dfrac{W_t}{\theta_t} - 1\right) \\
\dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right) \eta_{\pi,t} + \eta_{T,t} \psi \pi_t Y_t \\
\lim\limits_{t \to \infty} e^{-\rho t} \eta_{\pi,t} = 0 \\
\eta_{\pi,t}\Big|_{t=0} = 0
\end{cases}
\tag{73}
$$

7.5 Ramsey problem in TANK

This section shows how to solve the TANK optimal MP Ramsey problem using Calculus of Variations.

TANK model differs from the RANK model by adding a fraction τ of constrained hand-to-mouth households. In order to maintain comparability between HANK and TANK models, the constrained and unconstrained agents are different in their labor productivity and have different bond holdings. Labor productivity of the constrained household is calibrated to be the same as the average labor productivity of the constrained household in HANK model. Debt of the constrained household is equal to the borrowing constraint and the bond holdings of the unconstrained household are such that the zero supply of bonds is preserved.

Constrained household solves the problem of choosing labor and consumption at every moment, so there is only the labor supply equation and the budget constraint.

$$c_{k,t} = \lambda W_t l_{k,t} \varepsilon_k + d_t \varepsilon_k + T_t + r_t^b \underline{\mathrm{b}} \tag{74}$$

$$l_{k,t}^{\gamma} c_{k,t}^{\nu} = \frac{\lambda W_t \varepsilon_k}{\varphi} \tag{75}$$

Unconstrained household problem has intertemporal maximization

$$\mathcal{V}_t = \max_{\{c_u, l_u, \dot{b}_u\}_t} \mathbb{E}_t \int_0^{+\infty} e^{-\rho t} Z_t \left(\frac{c_{u,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{u,t}^{1+\gamma}}{1+\gamma} \right) dt \tag{76}$$

$$\text{s.t.} \quad c_{u,t} + \dot{b}_{u,t} = \lambda W_t l_{u,t} \varepsilon_u + d_t \varepsilon_u + T_t - r_t^b \frac{\tau}{1-\tau} \underline{\mathrm{b}} \tag{77}$$

Solving this problem under constant bond holding ($\dot{b}_{u,t} = 0$), gives the Euler equation and the labor supply equation.

$$r_t^b = \rho + \frac{\dot{Z}_t}{Z_t} - \nu \frac{\dot{c}_{u,t}}{c_{u,t}} \tag{78}$$

$$l_{u,t}^{\gamma} c_{u,t}^{\nu} = \frac{\lambda W_t \varepsilon_u}{\varphi} \tag{79}$$

Supply side stays the same, so the Phillips Curve is unchanged.

$$\left(r_t^b - \frac{\dot{Y}_t}{Y_t} \right) \pi_t = \frac{\phi_t - 1}{\psi} \left(\frac{\phi_t}{\phi_t - 1} m_t - 1 \right) + \dot{\pi}_t \tag{80}$$

Aggregate output is now produced with the labor of two agents

$$Y_t = \theta_t (\tau l_{k,t} \varepsilon_k + (1-\tau) l_{u,t} \varepsilon_u) \tag{81}$$

The Ramsey problem has the following form:

$$\mathcal{L}[c_u, c_k, l_u, l_k, W, Y, T, \pi, r^b] = \tag{82}$$

$$= \int_0^\infty e^{-\rho t} \left[\tau Z_t \left(\frac{c_{k,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{k,t}^{1+\gamma}}{1+\gamma} \right) + (1-\tau) Z_t \left(\frac{c_{u,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{u,t}^{1+\gamma}}{1+\gamma} \right) + \right. \tag{83}$$

$$+ \varrho_t \left(r_t^b - \rho + \frac{\dot{Z}_t}{Z_t} - \nu \frac{\dot{c}_{u,t}}{c_{u,t}} \right) + \mu_{k,t} \left(l_{k,t}^\gamma c_{k,t}^\nu - \frac{\lambda W_t \varepsilon_k}{\varphi} \right) + \mu_{u,t} \left(l_{u,t}^\gamma c_{u,t}^\nu - \frac{\lambda W_t \varepsilon_u}{\varphi} \right) + \tag{84}$$

$$+ \eta_{Y,t} \left(Y_t - \theta_t (\tau l_{k,t} \varepsilon_k + (1-\tau) l_{u,t} \varepsilon_u) \right) + \eta_{d,t} \left(d_t - \left(1 - \frac{\psi}{2} \pi_t^2 - \frac{W_t}{\theta_t} \right) \frac{Y_t}{\bar{\varepsilon}} \right) + \tag{85}$$

$$+ \eta_{k,t} \left(\lambda W_t l_{k,t} \varepsilon_k + T_t + d_t \varepsilon_k + r_t^b \underline{b} - c_{k,t} \right) + \tag{86}$$

$$+ \eta_{u,t} \left(\lambda W_t l_{u,t} \varepsilon_u + T_t + d_t \varepsilon_u - r_t^b \frac{\tau}{1-\tau} \underline{b} - c_{u,t} \right) + \eta_{T,t} \left(T_t - (1-\lambda) \frac{W_t}{\theta_t} Y_t \right) + \tag{87}$$

$$\left. + \eta_{\pi,t} \left(\frac{\phi_t - 1}{\psi} \left(\frac{\phi_t}{\phi_t - 1} \frac{W_t}{\theta_t} - 1 \right) + \dot{\pi}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t} \right) \pi_t \right) \right] dt \tag{88}$$

7.5.1 Control over consumption unconstrained

According to Calculus of Variations, I start with taking the first variation wrt the control function at hand. This implies taking the variation of the function and separately the variation of the variation of the derivative of this function. Notice that here I can take all the variations separately, as they do not interact between each other and the strong forms can be derived separately for each of the controls. This will not be the case in the HANK problem.

Weak form:

$$\frac{\delta \mathcal{L}}{\delta c_u} = \tag{89}$$

$$\int_0^\infty e^{-\rho t} \left[(1-\tau) Z_t c_{u,t}^{-\nu} v_{c_u,t} + \nu \frac{\varrho_t \dot{c}_{u,t}}{c_{u,t}^2} v_{c_u,t} - \nu \varrho_t \frac{\dot{v}_{c_u,t}}{c_{u,t}} + \nu \mu_{u,t} l_{u,t}^\gamma c_{u,t}^{\nu-1} v_{c_u,t} - \eta_{u,t} v_{c_u,t} \right] dt \tag{90}$$

To proceed further, I have to isolate the all the variations in their initial form without derivatives.

Rearranging the term with $\dot{v}_{c_u,t}$:

$$\int_0^\infty e^{-\rho t} \left[-\nu \varrho_t \frac{\dot{v}_{c_u,t}}{c_{u,t}} \right] dt = -e^{-\rho t} \nu \varrho_t \frac{v_{c_u,t}}{c_{u,t}} \Big|_0^\infty + \int_0^\infty e^{-\rho t} \left[\nu \frac{\dot{\varrho}_t}{c_{u,t}} - \nu \varrho_t \frac{\dot{c}_{u,t}}{c_{u,t}^2} - \rho \nu \frac{\varrho_t}{c_{u,t}} \right] v_{c_u,t} dt \tag{91}$$

Finally weak form is:

$$\frac{\delta \mathcal{L}}{\delta c_u} = \int_0^\infty e^{-\rho t} \left[(1-\tau) Z_t c_{u,t}^{-\nu} + \nu \frac{\dot{\varrho}_t}{c_{u,t}} - \rho \nu \frac{\varrho_t}{c_{u,t}} + \nu \mu_{u,t} l_{u,t}^\gamma c_{u,t}^{\nu-1} - \eta_{u,t} \right] v_{c_u,t} dt - \tag{92}$$

$$- e^{-\rho t} \nu \varrho_t \frac{v_{c_u,t}}{c_{u,t}} \Big|_0^\infty = 0 \tag{93}$$

Now using the standard CV approach to allow for any admissible variation, we can conclude that at any point the term that multiplies the variation has to be zero. Thus the strong form:

$$\begin{cases} \rho\varrho_t = \dot{\varrho}_t + (1-\tau)Z_t\dfrac{c_{u,t}^{1-\nu}}{\nu} + \mu_{u,t}l_t^{\gamma}c_{u,t}^{\nu} - \dfrac{\eta_{u,t}c_{u,t}}{\nu} \\ \lim\limits_{t\to\infty} e^{-\rho t}\nu\dfrac{\varrho_t}{c_{u,t}} = 0 \\ \varrho_t\Big|_{t=0} = 0 \end{cases} \tag{94}$$

7.5.2 Control over consumption constrained

Weak form:

$$\frac{\delta\mathcal{L}}{\delta c_k} = \int\limits_0^{\infty} e^{-\rho t}\left[\tau Z_t c_{k,t}^{-\nu}v_{c_k,t} + \nu\mu_{k,t}l_{k,t}^{\gamma}c_{k,t}^{\nu-1}v_{c_k,t} - \eta_{k,t}v_{c_k,t}\right]dt \tag{95}$$

Strong form:

$$\tau Z_t c_{k,t}^{-\nu} + \nu\mu_{k,t}l_{k,t}^{\gamma}c_{k,t}^{\nu-1} - \eta_{k,t} = 0 \tag{96}$$

7.5.3 Control over labor unconstrained

Weak form:

$$\frac{\delta\mathcal{L}}{\delta l_u} = \int\limits_0^{\infty} e^{-\rho t}\Big[-(1-\tau)Z_t\varphi l_{u,t}^{\gamma}v_{l_u,t} + \gamma\mu_{u,t}l_{u,t}^{\gamma-1}c_{u,t}^{\nu}v_{l_u,t} - \tag{97}$$

$$-\eta_{Y,t}\theta_t\varepsilon_u(1-\tau)v_{l_u,t} + \eta_{u,t}\lambda W_t\varepsilon_u v_{l_u,t}]\,dt = 0 \tag{98}$$

Strong form:

$$-(1-\tau)Z_t\varphi l_{u,t}^{\gamma} + \gamma\mu_{u,t}l_{u,t}^{\gamma-1}c_{u,t}^{\nu} - \eta_{Y,t}\theta_t\varepsilon_u(1-\tau) + \eta_{u,t}\lambda W_t\varepsilon_u = 0 \tag{99}$$

7.5.4 Control over labor constrained

Weak form:

$$\frac{\delta\mathcal{L}}{\delta l_k} = \int\limits_0^{\infty} e^{-\rho t}\left[-\tau Z_t\varphi l_{k,t}^{\gamma}v_{l_k,t} + \gamma\mu_{k,t}l_{k,t}^{\gamma-1}c_{k,t}^{\nu}v_{l_k,t} - \eta_{Y,t}\theta_t\varepsilon_k\tau v_{l_k,t} + \eta_{k,t}\lambda W_t\varepsilon_k v_{l_k,t}\right]dt = 0 \tag{100}$$

Strong form:

$$-\tau Z_t\varphi l_{k,t}^{\gamma} + \gamma\mu_{k,t}l_{k,t}^{\gamma-1}c_{k,t}^{\nu} - \eta_{Y,t}\theta_t\varepsilon_k\tau + \eta_{k,t}\lambda W_t\varepsilon_k = 0 \tag{101}$$

7.5.5 Control over wage

Weak form:

$$\frac{\delta\mathcal{L}}{\delta W} = \int_0^\infty e^{-\rho t}\left[-\mu_{k,t}\frac{\lambda\varepsilon_k}{\varphi} - \mu_{u,t}\frac{\lambda\varepsilon_u}{\varphi} - \eta_{T,t}\frac{(1-\lambda)}{\theta_t}Y_t + \right. \tag{102}$$

$$\left. +\eta_{d,t}\frac{Y_t}{\bar{\varepsilon}\theta_t} + \eta_{\pi,t}\frac{\phi_t}{\psi\theta_t} + \eta_{k,t}\lambda\varepsilon_k l_{k,t} + \eta_{u,t}\lambda\varepsilon_u l_{u,t}\right] v_{W,t}dt = 0 \tag{103}$$

Strong form:

$$-\mu_{k,t}\frac{\lambda\varepsilon_k}{\varphi} - \mu_{u,t}\frac{\lambda\varepsilon_u}{\varphi} - \eta_{T,t}\frac{(1-\lambda)}{\theta_t}Y_t + \eta_{d,t}\frac{Y_t}{\bar{\varepsilon}\theta_t} + \eta_{\pi,t}\frac{\phi_t}{\psi\theta_t} + \eta_{k,t}\lambda\varepsilon_k l_{k,t} + \eta_{u,t}\lambda\varepsilon_u l_{u,t} = 0 \tag{104}$$

7.5.6 Control over return

Weak form:

$$\frac{\delta\mathcal{L}}{\delta r^b} = \int_0^\infty e^{-\rho t}\left[\varrho_t v_{r^b,t} + \eta_{k,t}\underline{b}v_{r^b,t} - \eta_{u,t}\underline{b}\frac{\tau}{1-\tau}v_{r^b,t} - \eta_\pi\pi_t v_{r^b,t}\right]dt = 0 \tag{105}$$

Strong form:

$$\varrho_t = \eta_\pi\pi_t - \eta_{k,t}\underline{b} + \eta_{u,t}\underline{b}\frac{\tau}{1-\tau} \tag{106}$$

7.5.7 Control over output

Weak form:

$$\frac{\delta\mathcal{L}}{\delta Y} = \int_0^\infty e^{-\rho t}\left[-\eta_{d,t}\left(1-\frac{\psi}{2}\pi_t^2 - \frac{W_t}{\theta_t}\right)\frac{1}{\bar{\varepsilon}}v_{Y,t} - \right. \tag{107}$$

$$\left. -\eta_{T,t}(1-\lambda)\frac{W_t}{\theta_t}v_{Y,t} + \eta_{Y,t}v_{Y,t} + \eta_{\pi,t}\pi_t\frac{\dot{v}_{Y,t}}{Y_t} - \eta_{\pi,t}\pi_t\frac{\dot{Y}_t}{Y_t^2}v_{Y,t}\right]dt \tag{108}$$

In the same way as in the control over the consumption, to proceed further, I have to isolate all the variations in their initial form without derivatives.

Rearranging the term with $\dot{v}_{Y,t}$:

$$\int_0^\infty e^{-\rho t}\left[\eta_{\pi,t}\pi_t\frac{\dot{v}_{Y,t}}{Y_t}\right]dt = e^{-\rho t}\eta_{\pi,t}\pi_t\frac{v_{Y,t}}{Y_t}\Big|_0^\infty - \tag{109}$$

$$-\int_0^\infty e^{-\rho t}\left[\frac{\frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t)}{Y_t} - \frac{\eta_{\pi,t}\pi_t\dot{Y}_t}{Y_t^2} - \rho\frac{\eta_{\pi,t}\pi_t}{Y_t}\right]v_{Y,t}dt \tag{110}$$

This implies the weak form to be:

$$\frac{\delta\mathcal{L}}{\delta Y}=\int\limits_{0}^{\infty}e^{-\rho t}\left[-\eta_{T,t}\left(1-\frac{\psi}{2}\pi_t^2-\lambda\frac{W_t}{\theta_t}\right)+\eta_{Y,t}-\frac{\frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t)}{Y_t}+\rho\frac{\eta_{\pi,t}\pi_t}{Y_t}\right]v_{Y,t}dt+ \tag{111}$$

$$+e^{-\rho t}\eta_{\pi,t}\pi_t\frac{v_{Y,t}}{Y_t}\bigg|_0^{\infty}=0 \tag{112}$$

Strong form:

$$\begin{cases}\rho\eta_{\pi,t}\pi_t=\frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t)+\eta_{d,t}\left(1-\frac{\psi}{2}\pi_t^2-\frac{W_t}{\theta_t}\right)\frac{Y_t}{\overline{\varepsilon}}+\eta_{T,t}\left(1-\lambda\right)\frac{W_t}{\theta_t}Y_t-\eta_{Y,t}Y_t\\ \lim\limits_{t\to\infty}e^{-\rho t}\frac{\eta_{\pi,t}\pi_t}{Y_t}=0\\ \eta_{\pi,t}\pi_t\Big|_{t=0}=0\end{cases} \tag{113}$$

7.5.8 Control over inflation

Weak form:

$$\frac{\delta\mathcal{L}}{\delta\pi}=\int\limits_{0}^{\infty}e^{-\rho t}\left[-\eta_{d,t}\psi\pi_t\frac{Y_t}{\overline{\varepsilon}}v_{\pi,t}-\eta_{\pi,t}\left(r_t^b-\frac{\dot{Y}_t}{Y_t}-\rho\right)-\dot{\eta}_{\pi,t}v_{\pi,t}\right]dt+e^{-\rho t}\eta_{\pi,t}v_{\pi,t}\Big|_0^{\infty}=0 \tag{114}$$

Strong form:

$$\begin{cases}\dot{\eta}_{\pi,t}=\left(\rho-r_t^b+\frac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t}+\eta_{d,t}\psi\pi_t\frac{Y_t}{\overline{\varepsilon}}\\ \lim\limits_{t\to\infty}e^{-\rho t}\eta_{\pi,t}=0\\ \eta_{\pi,t}\Big|_{t=0}=0\end{cases} \tag{115}$$

7.5.9 Control over transfers

Weak form:

$$\frac{\delta\mathcal{L}}{\delta T}=\int\limits_{0}^{\infty}e^{-\rho t}\left[\eta_{T,t}v_{T,t}+\eta_{k,t}v_{T,t}+\eta_{u,t}v_{T,t}\right]dt=0 \tag{116}$$

Strong form:

$$\eta_{T,t}+\eta_{k,t}+\eta_{u,t}=0 \tag{117}$$

7.5.10 Control over dividend

Weak form:

$$\frac{\delta \mathcal{L}}{\delta d} = \int_0^\infty e^{-\rho t} \left[\eta_{d,t} v_{d,t} + \eta_{k,t}\varepsilon_k v_{d,t} + \eta_{u,t}\varepsilon_u v_{d,t}\right] dt = 0 \tag{118}$$

Strong form:

$$\eta_{d,t} + \eta_{k,t}\varepsilon_k + \eta_{u,t}\varepsilon_u = 0 \tag{119}$$

7.5.11 Optimal MP policy

Finally all conditions can be brought together in one system:

$$\begin{cases} \rho\varrho_t = \dot{\varrho}_t + (1-\tau)Z_t \dfrac{c_{u,t}^{1-\nu}}{\nu} + \mu_{u,t} l_t^\gamma c_{u,t}^\nu - \dfrac{\eta_{u,t} c_{u,t}}{\nu} \\ \lim\limits_{t\to\infty} e^{-\rho t}\nu \dfrac{\varrho_t}{c_{u,t}} = 0 \\ \varrho_t\Big|_{t=0} = 0 \\ \tau Z_t c_{k,t}^{-\nu} + \nu\mu_{k,t} l_{k,t}^\gamma c_{k,t}^{\nu-1} - \eta_{k,t} = 0 \\ -(1-\tau)Z_t\varphi l_{u,t}^\gamma + \gamma\mu_{u,t} l_{u,t}^{\gamma-1} c_{u,t}^\nu - \eta_{Y,t}\theta_t\varepsilon_u(1-\tau) + \eta_{u,t}\lambda W_t\varepsilon_u = 0 \\ -\tau Z_t\varphi l_{k,t}^\gamma + \gamma\mu_{k,t} l_{k,t}^{\gamma-1} c_{k,t}^\nu - \eta_{Y,t}\theta_t\varepsilon_k\tau + \eta_{k,t}\lambda W_t\varepsilon_k = 0 \\ -\mu_{k,t}\dfrac{\lambda\varepsilon_k}{\varphi} - \mu_{u,t}\dfrac{\lambda\varepsilon_u}{\varphi} - \eta_{T,t}\dfrac{(1-\lambda)}{\theta_t} Y_t + \eta_{d,t}\dfrac{Y_t}{\overline{\varepsilon}\theta_t} + \eta_{\pi,t}\dfrac{\phi_t}{\psi\theta_t} + \eta_{k,t}\lambda\varepsilon_k l_{k,t} + \eta_{u,t}\lambda\varepsilon_u l_{u,t} = 0 \\ \varrho_t = \eta_\pi\pi_t - \eta_{k,t}\underline{b} + \eta_{u,t}\underline{b}\frac{\tau}{1-\tau} \\ \rho\eta_{\pi,t}\pi_t = \frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t) + \eta_{d,t}\left(1 - \dfrac{\psi}{2}\pi_t^2 - \dfrac{W_t}{\theta_t}\right)\dfrac{Y_t}{\overline{\varepsilon}} + \eta_{T,t}(1-\lambda)\dfrac{W_t}{\theta_t} Y_t - \eta_{Y,t} Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\dfrac{\eta_{\pi,t}\pi_t}{Y_t} = 0 \\ \eta_{\pi,t}\pi_t\Big|_{t=0} = 0 \\ \dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{d,t}\psi\pi_t\dfrac{Y_t}{\overline{\varepsilon}} \\ \lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\ \eta_{\pi,t}\Big|_{t=0} = 0 \\ \eta_{T,t} + \eta_{k,t} + \eta_{u,t} = 0 \\ \eta_{d,t} + \eta_{k,t}\varepsilon_k + \eta_{u,t}\varepsilon_u = 0 \end{cases} \tag{120}$$

Now equations can be rearranged to get the co-state variables updates (Combining control over output, control over inflation, and PC in the same way as in RANK):

$$
\begin{cases}
\gamma\mu_{u,t}l_{u,t}^{\gamma-1}c_{u,t}^{\nu}-(1-\tau)Z_t\varphi l_{u,t}^{\gamma}-\eta_{Y,t}\theta_t\varepsilon_u(1-\tau)+\eta_{u,t}\lambda W_t\varepsilon_u=0\\
\gamma\mu_{k,t}l_{k,t}^{\gamma-1}c_{k,t}^{\nu}-\tau Z_t\varphi l_{k,t}^{\gamma}-\eta_{Y,t}\theta_t\varepsilon_k\tau+\eta_{k,t}\lambda W_t\varepsilon_k=0\\
\eta_{k,t}=\tau Z_t c_{k,t}^{-\nu}+\mu_{k,t}\nu l_{k,t}^{\gamma}c_{k,t}^{\nu-1}\\
\eta_{T,t}=-\eta_{k,t}-\eta_{u,t}\\
\eta_{d,t}=-\eta_{k,t}\varepsilon_k-\eta_{u,t}\varepsilon_u\\
\\
\eta_{Y,t}=\eta_{d,t}\left(1+\frac{\psi}{2}\pi_t^2-\frac{W_t}{\theta_t}\right)\frac{1}{\overline{\varepsilon}}-\eta_{\pi,t}\frac{1}{Y_t}\frac{\phi_t-1}{\psi}\left(\frac{\phi_t}{\phi_t-1}\frac{W_t}{\theta_t}-1\right)\\
\varrho_t=\eta_\pi\pi_t-\eta_{k,t}\underline{b}+\eta_{u,t}\underline{b}\frac{\tau}{1-\tau}\\
\eta_{\pi,t}\frac{\phi_t}{\psi\theta_t}-\mu_{k,t}\frac{\lambda\varepsilon_k}{\varphi}-\mu_{u,t}\frac{\lambda\varepsilon_u}{\varphi}-\eta_{T,t}\frac{(1-\lambda)}{\theta_t}Y_t+\eta_{d,t}\frac{Y_t}{\overline{\varepsilon}\theta_t}+\eta_{k,t}\lambda l_{k,t}\varepsilon_k+\eta_{u,t}\lambda l_{u,t}\varepsilon_u=0\\
\rho\varrho_t=\dot{\varrho}_t+(1-\tau)Z_t\frac{c_{u,t}^{1-\nu}}{\nu}+\mu_{u,t}l_t^{\gamma}c_{u,t}^{\nu}-\frac{\eta_{u,t}c_{u,t}}{\nu}\\
\lim_{t\to\infty}e^{-\rho t}\nu\frac{\varrho_t}{c_{u,t}}=0\\
\varrho_t\Big|_{t=0}=0\\
\dot{\eta}_{\pi,t}=\left(\rho-r_t^b+\frac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t}+\eta_{d,t}\psi\pi_t\frac{Y_t}{\overline{\varepsilon}}\\
\lim_{t\to\infty}e^{-\rho t}\eta_{\pi,t}=0\\
\eta_{\pi,t}\Big|_{t=0}=0
\end{cases}
\tag{121}
$$

7.6 Ramsey problem in HANK

This section provides comprehensive scratch notes for the Ramsey problem solution using Calculus of Variations.

First, let's write the problem of the Ramsey planner:

$$\mathcal{L}[f,\mathcal{V},c,l,W,Y,\pi,r^b,T,d]= \tag{122}$$

$$=\int_0^\infty e^{-\rho t}\left[\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu}-\varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma},f_{i,t}\right\rangle+\left\langle \zeta_{i,t},\mathcal{A}^*_{i,t}f_{i,t}-\frac{\partial f_{i,t}}{\partial t}\right\rangle\right. \tag{123}$$

$$+\left\langle \varrho_{i,t},\frac{c_{i,t}^{1-\nu}}{1-\nu}-\varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma}+\mathcal{A}_{i,t}\mathcal{V}_{i,t}+\frac{\partial\mathcal{V}_{i,t}}{\partial t}-\rho\mathcal{V}_{i,t}\right\rangle \tag{124}$$

$$+\left\langle \mu_{i,t},c_{i,t}^{-\nu}-\frac{\partial\mathcal{V}_{i,t}}{\partial b}\right\rangle+\left\langle \kappa_{i,t},l_{i,t}^{\gamma}c_{i,t}^{\nu}-\frac{\lambda W_t\varepsilon_{i,t}}{\varphi}\right\rangle \tag{125}$$

$$+\eta_{b,t}\left\langle b_{i,t},f_{i,t}\right\rangle+\eta_{Y,t}\left(Y_t-\theta_t\left\langle l_{i,t}\varepsilon_{i,t},f_{i,t}\right\rangle\right) \tag{126}$$

$$+\eta_{T,t}\left(T_t-(1-\lambda)\frac{W_t}{\theta_t}Y_t\right)+\eta_{d,t}\left(d_t-\left(1-\frac{\psi}{2}\pi_t^2-\frac{W_t}{\theta_t}\right)\frac{Y_t}{\bar{\varepsilon}}\right) \tag{127}$$

$$\left.+\eta_{\pi,t}\left(\frac{\phi-1}{\psi}\left(\frac{\phi}{\phi-1}\frac{W_t}{\theta_t}-1\right)+\dot{\pi}_t-\left(r_t^b-\frac{\dot{Y}_t}{Y_t}\right)\pi_t\right)\right]dt \tag{128}$$

Where

$$\mathcal{A}_{i,t}\mathcal{V}_{i,t}=\dot{b}_{i,t}\frac{\partial\mathcal{V}_{i,t}}{\partial b}+\rho_\varepsilon\varepsilon_{i,t}(\bar{e}-e_{i,t})\frac{\partial\mathcal{V}_{i,t}}{\partial\varepsilon}+\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial^2\mathcal{V}_{i,t}}{\partial\varepsilon^2} \tag{129}$$

$$\mathcal{A}^*_{i,t}f_{i,t}=-\frac{\partial}{\partial b}\dot{b}_{i,t}f_{i,t}-\frac{\partial}{\partial\varepsilon}\rho_\varepsilon\varepsilon_{i,t}(\bar{e}-e_{i,t})f_{i,t}+\frac{\partial^2}{\partial\varepsilon^2}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}f_{i,t} \tag{130}$$

$$\dot{b}_{i,t}=\lambda W_t l_{i,t}\varepsilon_{i,t}+r_t^b b_{i,t}+T_t+d_t\varepsilon_{i,t}-c_{i,t} \tag{131}$$

And for $\mathcal{E},\mathcal{B}$ defined as sets such that $\varepsilon\in\mathcal{E},b\in\mathcal{B}$ and $\partial\mathcal{E},\partial\mathcal{B}$ defined as boundaries of sets $\mathcal{E},\mathcal{B}$.

$$\langle\cdot,\cdot\rangle\equiv\int_{\mathcal{E}}\int_{\mathcal{B}}\cdot\,\cdot\;dbd\varepsilon \tag{132}$$

Using calculus of variations I will take the functional derivatives of $\mathcal{L}$ wrt all of the control functions. Notice that the key idea is to treat derivatives of the control functions in the Lagrangean as functions themselves.

One important aspect is that the variations should be taken all together, and not separately. This matters for the treatment of the boundary conditions. For the readability, I will write the variations one by one, but highlight the place in the derivation where I use the fact that variations are taken together to treat the boundary constraint on the bonds.

7.6.1 Control over distribution

Weak form:

$$\frac{\delta \mathcal{L}}{\delta f} = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma}, v_{i,t} \right\rangle + \left\langle \zeta_{i,t}, \mathcal{A}_{i,t}^{*} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right\rangle \right. \tag{133}$$

$$\left. + \eta_{b,t} \left\langle b_{i,t}, v_{i,t} \right\rangle + \eta_{Y,t} \left(-\theta_t \left\langle l_{i,t} \varepsilon_{i,t}, v_{i,t} \right\rangle \right) \right] dt = 0 \tag{134}$$

Now to move to the strong form some rearrangement should be done to the differential part of the weak form

$$\int_0^\infty e^{-\rho t} \left[\left\langle \zeta_{i,t}, \mathcal{A}_{i,t}^{*} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right\rangle \right] dt = \tag{135}$$

$$\int_0^\infty e^{-\rho t} \left[\int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(\mathcal{A}_{i,t}^{*} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right) db d\varepsilon \right] dt = \tag{136}$$

$$\int_0^\infty e^{-\rho t} \left[\int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(-\frac{\partial}{\partial b} \dot{b}_{i,t} v_{i,t} - \frac{\partial}{\partial \varepsilon} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} + \frac{\partial^2}{\partial \varepsilon^2} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right) db d\varepsilon \right] dt \tag{137}$$

Treating all four parts of the integral separately:

First:

$$-\int_0^\infty e^{-\rho t} \left[\int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(\frac{\partial}{\partial b} \dot{b}_{i,t} v_{i,t} \right) db d\varepsilon \right] dt = \tag{138}$$

$$-\int_0^\infty e^{-\rho t} \int_{\mathcal{E}} \left[\left(\zeta_{i,t} \dot{b}_{i,t} v_{i,t} \right) \Big|_{\partial \mathcal{B}} - \int_{\mathcal{B}} \dot{b}_{i,t} v_{i,t} \frac{\partial \zeta_{i,t}}{\partial b} db \right] d\varepsilon dt \tag{139}$$

Second:

$$-\int_0^\infty e^{-\rho t} \left[\int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(\frac{\partial}{\partial \varepsilon} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \right) db d\varepsilon \right] dt = \tag{140}$$

$$-\int_0^\infty e^{-\rho t} \int_{\mathcal{B}} \left[(\zeta_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t}) \Big|_{\partial \mathcal{E}} - \int_{\mathcal{E}} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \frac{\partial \zeta_{i,t}}{\partial \varepsilon} d\varepsilon \right] db dt \tag{141}$$

Third:

$$\int_0^\infty e^{-\rho t}\left[\int_{\mathcal{E}}\int_{\mathcal{B}}\zeta_{i,t}\left(\frac{\partial^2}{\partial\varepsilon^2}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)dbd\varepsilon\right]dt = \tag{142}$$

$$\int_0^\infty e^{-\rho t}\int_{\mathcal{B}}\left[\zeta_{i,t}\left(\frac{\partial}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)\Bigg|_{\partial\mathcal{E}} - \int_{\mathcal{E}}\frac{\partial\zeta_{i,t}}{\partial\varepsilon}\left(\frac{\partial}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)d\varepsilon\right]dbdt = \tag{143}$$

$$\int_0^\infty e^{-\rho t}\int_{\mathcal{B}}\left[\zeta_{i,t}\left(\frac{\partial}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)\Bigg|_{\partial\mathcal{E}} - \left(\frac{\partial\zeta_{i,t}}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)\Bigg|_{\partial\mathcal{E}} + \int_{\mathcal{E}}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\frac{\partial^2\zeta_{i,t}}{\partial\varepsilon^2}d\varepsilon\right]dbdt \tag{144}$$

Fourth:

$$-\int_0^\infty e^{-\rho t}\left[\int_{\mathcal{E}}\int_{\mathcal{B}}\zeta_{i,t}\left(\frac{\partial v_{i,t}}{\partial t}\right)dbd\varepsilon\right]dt = -\int_{\mathcal{E}}\int_{\mathcal{B}}\left[\int_0^\infty e^{-\rho t}\zeta_{i,t}\frac{\partial v_{i,t}}{\partial t}dt\right]dbd\varepsilon = \tag{145}$$

$$-\int_{\mathcal{E}}\int_{\mathcal{B}}\left[e^{-\rho t}\zeta_{i,t}v_{i,t}\Bigg|_0^\infty - \int_0^\infty v_{i,t}\frac{\partial}{\partial t}e^{-\rho t}\zeta_{i,t}dt\right]dbd\varepsilon = \tag{146}$$

$$-\int_{\mathcal{E}}\int_{\mathcal{B}}\left[e^{-\rho t}\zeta_{i,t}v_{i,t}\Bigg|_0^\infty - \int_0^\infty v_{i,t}e^{-\rho t}\left(\frac{\partial\zeta_{i,t}}{\partial t}-\rho\zeta_{i,t}\right)dt\right]dbd\varepsilon \tag{147}$$

Finally, notice that the boundary constraint on $f_{i,t}$ implies that it has to be equal to zero on any boundary except $b = \underline{b}$ for values of $\dot{b} = 0$, as well as $\dfrac{\partial f_{i,t}}{\partial\varepsilon} = 0$ for $f \in \partial\mathcal{E}$. This implies that all the boundaries $\Big|_{\partial\mathcal{B}}$ and $\Big|_{\partial\mathcal{E}}$ actually cancel out. Moreover, since distribution is fixed at $t = 0$, the only boundary condition that is left is the one at $t \to \infty$. Then the term can be written as

$$\int_0^\infty e^{-\rho t}\left[\left\langle\zeta_{i,t}, \mathcal{A}^*_{i,t}v_{i,t} - \frac{\partial v_{i,t}}{\partial t}\right\rangle\right]dt = \tag{148}$$

$$\int_0^\infty e^{-\rho t}\left[\left\langle\mathcal{A}_{i,t}\zeta_{i,t} + \frac{\partial\zeta_{i,t}}{\partial t} - \rho\zeta_{i,t}, v_{i,t}\right\rangle\right]dt - \lim_{t\to\infty}\int_{\mathcal{E}}\int_{\mathcal{B}}e^{-\rho t}\zeta_{i,t}v_{i,t}dbd\varepsilon \tag{149}$$

Strong form:

$$\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t}\zeta_{i,t} + \frac{\partial\zeta_{i,t}}{\partial t} - \rho\zeta_{i,t} + \eta_{b,t}b_{i,t} - \eta_{Y,t}\theta_t l_{i,t}\varepsilon_{i,t} = 0 \tag{150}$$

$$\lim_{t\to\infty} e^{-\rho t}\zeta_{i,t} = 0 \tag{151}$$

7.6.2 Control over household value

Weak form:

$$\frac{\delta \mathcal{L}}{\delta \mathcal{V}} = \int_0^\infty e^{-\rho t} \left[\left\langle \varrho_{i,t}, \mathcal{A}_{i,t} v_{i,t} + \frac{\partial v_{i,t}}{\partial t} - \rho v_{i,t} \right\rangle + \left\langle \mu_{i,t}, -\frac{\partial v_{i,t}}{\partial b} \right\rangle \right] dt = 0 \tag{152}$$

Where

$$\mathcal{A}_{i,t} v_{i,t} = \dot{b}_{i,t} \frac{\partial v_{i,t}}{\partial b} + \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) \frac{\partial v_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial^2 v_{i,t}}{\partial \varepsilon^2} \tag{153}$$

$$\dot{b}_{i,t} = \lambda W_t l_{i,t} \varepsilon_{i,t} + r_t^b b_{i,t} + T_t + d_t s_i - c_{i,t} \tag{154}$$

For $\mathcal{A}_{i,t} v_{i,t}$ the following rearrangement can be done:

$$\langle \varrho_{i,t}, \mathcal{A}_{i,t} v_{i,t} \rangle = \int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \mathcal{A}_{i,t} v_{i,t} db d\varepsilon = \tag{155}$$

$$\int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \left(\dot{b}_{i,t} \frac{\partial v_{i,t}}{\partial b} + \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) \frac{\partial v_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial^2 v_{i,t}}{\partial \varepsilon^2} \right) db d\varepsilon \tag{156}$$

As in the previous case, working the three terms separately.
First:

$$\int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \dot{b}_{i,t} \frac{\partial v_{i,t}}{\partial b} db d\varepsilon = \int_{\mathcal{E}} \varrho_{i,t} \dot{b}_{i,t} v_{i,t} \Big|_{\partial \mathcal{B}} d\varepsilon - \int_{\mathcal{E}} \int_{\mathcal{B}} v_{i,t} \frac{\partial}{\partial b} \varrho_{i,t} \dot{b}_{i,t} db d\varepsilon \tag{157}$$

Second:

$$\int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) \frac{\partial v_{i,t}}{\partial \varepsilon} db d\varepsilon = \tag{158}$$

$$\int_{\mathcal{B}} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \Big|_{\partial \mathcal{E}} db - \int_{\mathcal{B}} \int_{\mathcal{E}} v_{i,t} \frac{\partial}{\partial \varepsilon} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) d\varepsilon db \tag{159}$$

Third:

$$\int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial^2 v_{i,t}}{\partial \varepsilon^2} db d\varepsilon = \tag{160}$$

$$\int_{\mathcal{B}} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial v_{i,t}}{\partial \varepsilon} \Big|_{\partial \mathcal{E}} db - \int_{\mathcal{B}} \int_{\mathcal{E}} \frac{\partial v_{i,t}}{\partial \varepsilon} \frac{\partial}{\partial \varepsilon} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} d\varepsilon db = \tag{161}$$

$$\int_{\mathcal{B}} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial v_{i,t}}{\partial \varepsilon} \Big|_{\partial \mathcal{E}} db - \int_{\mathcal{B}} v_{i,t} \frac{\partial}{\partial \varepsilon} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \Big|_{\partial \mathcal{E}} db + \int_{\mathcal{B}} \int_{\mathcal{E}} v_{i,t} \frac{\partial^2}{\partial \varepsilon^2} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} d\varepsilon db \tag{162}$$

Notice, that because of Inada conditions, the values of $\frac{\partial \mathcal{V}}{\partial b}$ and $\frac{\partial \mathcal{V}}{\partial \varepsilon}$ are fixed to be zero for

$b \to \infty$ and $\varepsilon \to \infty$. This implies that variation of partial derivatives has to be zero at these boundaries as well. This means that the expression becomes

$$\langle \varrho_{i,t}, \mathcal{A}_{i,t} v_{i,t} \rangle = \langle \mathcal{A}^*_{i,t} \varrho_{i,t}, v_{i,t} \rangle + \int\limits_{\mathcal{E}} \varrho_{i,t} \dot{b}_{i,t} v_{i,t} \Big|_{b=\underline{b}} d\varepsilon + \int\limits_{\mathcal{B}} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \Big|_{\varepsilon=\underline{\varepsilon}} db+ \tag{163}$$

$$+ \int\limits_{\mathcal{B}} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma^2_\varepsilon}{2} \frac{\partial v_{i,t}}{\partial \varepsilon} \Big|_{\varepsilon=\underline{\varepsilon}} db - \int\limits_{\mathcal{B}} v_{i,t} \frac{\partial}{\partial \varepsilon} \varrho_{i,t} \varepsilon_{i,t} \frac{\sigma^2_\varepsilon}{2} \Big|_{\varepsilon=\underline{\varepsilon}} db \tag{164}$$

Finally notice, that since the distribution of ε is a log-normal, then $\underline{\varepsilon} = 0$, so that almost all the terms cancel out

$$\langle \varrho_{i,t}, \mathcal{A}_{i,t} v_{i,t} \rangle = \langle \mathcal{A}^*_{i,t} \varrho_{i,t}, v_{i,t} \rangle + \int\limits_{\mathcal{E}} \varrho_{i,t} \dot{b}_{i,t} v_{i,t} \Big|_{b=\underline{b}} d\varepsilon - \int\limits_{\mathcal{B}} v_{i,t} \varrho_{i,t} \frac{\sigma^2_\varepsilon}{2} \Big|_{\varepsilon=\underline{\varepsilon}} db \tag{165}$$

For the FOC constraints (utilizing Inada conditions on $b \to \infty$):

$$-\left\langle \mu_{i,t}, \frac{\partial v_{i,t}}{\partial b} \right\rangle = -\int\limits_{\mathcal{E}} \mu_{i,t} v_{i,t} \Big|_{b=\underline{b}} d\varepsilon + \int\limits_{\mathcal{E}} \int\limits_{\mathcal{B}} v_{i,t} \frac{\partial \mu_{i,t}}{\partial b} db d\varepsilon \tag{166}$$

For the time derivative:

$$\int\limits_0^\infty e^{-\rho t} \left[\left\langle \varrho_{i,t}, \frac{\partial v_{i,t}}{\partial t} - \rho v_{i,t} \right\rangle \right] dt = \left\langle \varrho_{i,t} e^{-\rho t} v_{i,t} \Big|_0^\infty \right\rangle - \left\langle \int\limits_0^\infty e^{-\rho t} v_{i,t} \frac{\partial \varrho_{i,t}}{\partial t} dt \right\rangle \tag{167}$$

Finally, this means that the strong form is:

$$\mathcal{A}^*_{i,t} \varrho_{i,t} - \frac{\partial \varrho_{i,t}}{\partial t} + \frac{\partial \mu_{i,t}}{\partial b} = 0 \tag{168}$$

$$\varrho_{i,0} = 0 \tag{169}$$

$$\varrho_{i,t} \Big|_{\varepsilon=\underline{\varepsilon}} = 0 \tag{170}$$

$$\varrho_{i,t} \dot{b}_{i,t} - \mu_{i,t} \Big|_{b=\underline{b}} = 0 \tag{171}$$

7.6.3 Control over household consumption

Weak form:

$$\frac{\delta \mathcal{L}}{\delta c} = \int\limits_0^\infty e^{-\rho t} \Bigg[\left\langle c^{-\nu}_{i,t} v_{i,t}, f_{i,t} \right\rangle + \left\langle \zeta_{i,t}, \frac{\partial}{\partial b} v_{i,t} f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, c^{-\nu}_{i,t} v_{i,t} - \frac{\partial \mathcal{V}_{i,t}}{\partial b} v_{i,t} \right\rangle \tag{172}$$

$$+ \left\langle \mu_{i,t}, -\nu c^{-\nu-1}_{i,t} v_{i,t} \right\rangle + \left\langle \kappa_{i,t}, \nu l^\gamma_{i,t} c^{\nu-1}_{i,t} v_{i,t} \right\rangle \Bigg] dt = 0 \tag{173}$$

Rearranging the term with the derivative

$$\int_0^\infty e^{-\rho t}\left[\left\langle \zeta_{i,t}, \frac{\partial}{\partial b} v_{i,t} f_{i,t}\right\rangle\right] dt = \int_0^\infty e^{-\rho t}\left[\int_{\mathcal{E}}\int_{\mathcal{B}} \zeta_{i,t}\frac{\partial}{\partial b} v_{i,t} f_{i,t} db d\varepsilon\right] dt = \tag{174}$$

$$\int_0^\infty e^{-\rho t}\int_{\mathcal{E}}\left[\zeta_{i,t} v_{i,t} f_{i,t}\Big|_{\partial\mathcal{B}} - \int_{\mathcal{B}} v_{i,t} f_{i,t}\frac{\partial \zeta_{i,t}}{\partial b} db\right] d\varepsilon dt \tag{175}$$

The strong form is:

$$c_{i,t}^{-\nu} f_{i,t} - f_{i,t}\frac{\partial \zeta_{i,t}}{\partial b} + \varrho_{i,t}\left(c_{i,t}^{-\nu} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}\right) - \mu_{i,t}\nu c_{i,t}^{-\nu-1} + \kappa_{i,t}\nu l_{i,t}^{\gamma} c_{i,t}^{\nu-1} = 0 \tag{176}$$

$$\zeta_{i,t} f_{i,t}\Big|_{b=\underline{b}} = 0 \tag{177}$$

Since FOC wrt c holds, the condition becomes

$$\left(c_{i,t}^{-\nu} - \frac{\partial \zeta_{i,t}}{\partial b}\right) f_{i,t} - \mu_{i,t}\nu c_{i,t}^{-\nu-1} + \kappa_{i,t}\nu l_{i,t}^{\gamma} c_{i,t}^{\nu-1} = 0 \tag{178}$$

$$\zeta_{i,t} f_{i,t}\Big|_{b=\underline{b}} = 0 \tag{179}$$

Here I come back to the point that that the variations should be taken together. This means that the boundary conditions for $b = \underline{b}$ should be satisfied as a sum, not separately from each other. This means that the following should hold:

Combining the equation for μ with constraint from value control one can get the following conditions:

If $\dot{b}_{i,t} \neq 0$ at $b = \underline{b}$, then $f_{i,t} = 0$

$$\Rightarrow \kappa_{i,t} = 0 \Rightarrow \mu_{i,t} = 0 \Rightarrow \varrho_{i,t} = 0 \tag{180}$$

If $\dot{b}_{i,t} = 0$ at $b = \underline{b}$, then $f_{i,t} \neq 0 \Rightarrow \mu_{i,t} = 0$. Using FOC l from below:

$$\Rightarrow \left(\frac{c_{i,t}}{\nu} + \frac{l_{i,t}}{\gamma}\lambda W_t \varepsilon_{i,t}\right)\left(\frac{\partial \zeta_{i,t}}{\partial b} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}\right) = \frac{\eta_{Y,t}\theta_t \varepsilon_{i,t} l_{i,t}}{\gamma} \tag{181}$$

7.6.4 Control over household labor

Weak form:

$$\frac{\delta \mathcal{L}}{\delta l} = \int_0^\infty e^{-\rho t}\left[\left\langle -\varphi l_{i,t}^{\gamma} v_{i,t}, f_{i,t}\right\rangle + \left\langle \zeta_{i,t}, -\frac{\partial}{\partial b}\lambda W_t v_{i,t}\varepsilon_{i,t} f_{i,t}\right\rangle\right. \tag{182}$$

$$+ \left\langle \varrho_{i,t}, -\varphi l_{i,t}^{\gamma} v_{i,t} + \lambda W_t v_{i,t}\varepsilon_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial b}\right\rangle \tag{183}$$

$$\left. + \left\langle \kappa_{i,t}, \gamma l_{i,t}^{\gamma-1} v_{i,t} c_{i,t}^{\nu}\right\rangle + \eta_{Y,t}\left(-\theta_t\left\langle v_{i,t}\varepsilon_{i,t}, f_{i,t}\right\rangle\right)\right] dt = 0 \tag{184}$$

After rearranging, I get the strong form:

$$\left(-\varphi l_{i,t}^{\gamma}+\lambda W_t\varepsilon_{i,t}\frac{\partial\zeta_{i,t}}{\partial b}\right)f_{i,t}+\kappa_{i,t}\gamma l_{i,t}^{\gamma-1}c_{i,t}^{\nu}-\eta_{Y,t}\theta_t\varepsilon_{i,t}f_{i,t}=0 \tag{185}$$

7.6.5 Control over wage

Weak form:

$$\frac{\delta\mathcal{L}}{\delta W}=\int_0^{\infty}e^{-\rho t}\Bigg[\left\langle\zeta_{i,t},-\frac{\partial}{\partial b}\lambda v_t l_{i,t}\varepsilon_{i,t}f_{i,t}\right\rangle+\left\langle\varrho_{i,t},\lambda v_t l_{i,t}\varepsilon_{i,t}\frac{\partial\mathcal{V}_{i,t}}{\partial b}\right\rangle \tag{186}$$

$$+\left\langle\kappa_{i,t},-\frac{\lambda v_t\varepsilon_{i,t}}{\varphi}\right\rangle-\eta_{T,t}\left(1-\lambda\right)\frac{v_t}{\theta_t}Y_t+\eta_{d,t}\frac{v_t}{\theta_t}\varepsilon_{i,t}+\eta_{\pi,t}\left(\frac{\phi}{\psi}\frac{v_t}{\theta_t}\right)\Bigg]dt=0 \tag{187}$$

Strong form:

$$\left\langle\frac{\partial\zeta_{i,t}}{\partial b},\lambda l_{i,t}\varepsilon_{i,t}f_{i,t}\right\rangle+\left\langle\varrho_{i,t}\lambda l_{i,t}\varepsilon_{i,t},\frac{\partial\mathcal{V}_{i,t}}{\partial b}\right\rangle-\left\langle\kappa_{i,t},\frac{\lambda\varepsilon_{i,t}}{\varphi}\right\rangle- \tag{188}$$

$$-\eta_{T,t}\left(1-\lambda\right)\frac{Y_t}{\overline{\varepsilon}\theta_t}+\eta_{d,t}\frac{Y_t}{\theta_t}+\eta_{\pi,t}\frac{\phi}{\psi\theta_t}=0 \tag{189}$$

7.6.6 Control over output

Weak form:

$$\frac{\delta\mathcal{L}}{\delta Y}=\int_0^{\infty}e^{-\rho t}\Bigg[\eta_{Y,t}v_t-\eta_{T,t}\left(1-\lambda\right)\frac{W_t}{\theta_t}v_t-\eta_{d,t}\left(1-\frac{\psi}{2}\pi_t^2-\frac{W_t}{\theta_t}\right)\frac{1}{\overline{\varepsilon}}v_t+ \tag{190}$$

$$+\eta_{\pi,t}\frac{\dot{v}_t}{Y_t}\pi_t-\eta_{\pi,t}\frac{\dot{Y}_t}{Y_t}\frac{v_t}{Y_t}\pi_t\Bigg]dt=0 \tag{191}$$

Rearranging time variation:

$$\int_0^{\infty}e^{-\rho t}\eta_{\pi,t}\frac{\dot{v}_t}{Y_t}\pi_t dt=e^{-\rho t}\eta_{\pi,t}\frac{v_t}{Y_t}\pi_t\Bigg|_0^{\infty}-\int_0^{\infty}v_t\frac{\partial}{\partial t}e^{-\rho t}\eta_{\pi,t}\frac{1}{Y_t}\pi_t dt \tag{192}$$

Strong form:

$$\eta_{Y,t}-\eta_{T,t}\left(1-\lambda\right)\frac{W_t}{\theta_t}-\eta_{d,t}\left(1-\frac{\psi}{2}\pi_t^2-\frac{W_t}{\theta_t}\right)\frac{1}{\overline{\varepsilon}}-\eta_{\pi,t}\frac{\dot{Y}_t}{Y_t}\frac{1}{Y_t}\pi_t- \tag{193}$$

$$-\frac{\partial}{\partial t}\left(\eta_{\pi,t}\frac{1}{Y_t}\pi_t\right)+\rho\left(\eta_{\pi,t}\frac{1}{Y_t}\pi_t\right)=0 \tag{194}$$

$$\eta_{\pi,0}\pi_0=0 \tag{195}$$

Which can be rearranged to

$$\eta_{Y,t} - \eta_{T,t}(1-\lambda)\frac{W_t}{\theta_t} - \eta_{d,t}\left(1 - \frac{\psi}{2}\pi_t^2 - \frac{W_t}{\theta_t}\right)\frac{1}{\overline{\varepsilon}} - \left(\frac{\dot{\eta}_{\pi,t}}{\eta_{\pi,t}} + \frac{\dot{\pi}_t}{\pi_t} - \rho\right)\frac{\eta_{\pi,t}\pi_t}{Y_t} = 0 \tag{196}$$

$$\eta_{\pi,0}\pi_0 = 0 \tag{197}$$

7.6.7 Control over inflation

Weak form:

$$\frac{\delta\mathcal{L}}{\delta\pi} = \int_0^\infty e^{-\rho t}\left[\eta_{d,t}\psi\pi_t v_t \frac{Y_t}{\overline{\varepsilon}} + \eta_{\pi,t}\left(\dot{v}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)v_t\right)\right]dt = 0 \tag{198}$$

$$\int_0^\infty e^{-\rho t}\left[\eta_{d,t}\psi\pi_t v_t \frac{Y_t}{\overline{\varepsilon}} - \eta_{\pi,t}\left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)v_t - \dot{\eta}_{\pi,t}v_t + \rho\eta_{\pi,t}v_t\right]dt + e^{-\rho t}\eta_{\pi,t}v_t\Big|_0^\infty = 0 \tag{199}$$

Strong form:

$$\eta_{d,t}\psi\pi_t\frac{Y_t}{\overline{\varepsilon}} - \eta_{\pi,t}\left(r_t^b - \frac{\dot{Y}_t}{Y_t} - \rho\right) - \dot{\eta}_{\pi,t} = 0 \tag{200}$$

$$\eta_{\pi,0} = 0 \tag{201}$$

7.6.8 Control over real return

Weak form:

$$\frac{\delta\mathcal{L}}{\delta r^b} = \int_0^\infty e^{-\rho t}\left[\left\langle\frac{\partial\zeta_{i,t}}{\partial b}, v_t b_{i,t} f_{i,t}\right\rangle + \left\langle\varrho_{i,t}, v_t b_{i,t}\frac{\partial\mathcal{V}_{i,t}}{\partial b}\right\rangle - \eta_{\pi,t}v_t\pi_t\right]dt = 0 \tag{202}$$

$$\zeta_{i,t}f_{i,t}b_{i,t}\Big|_{b=\underline{b}} = 0 \tag{203}$$

Strong form:

$$\left\langle\frac{\partial\zeta_{i,t}}{\partial b}, b_{i,t}f_{i,t}\right\rangle + \left\langle\varrho_{i,t}, b_{i,t}\frac{\partial\mathcal{V}_{i,t}}{\partial b}\right\rangle - \eta_{\pi,t}\pi_t = 0 \tag{204}$$

7.6.9 Control over transfers

Weak form:

$$\frac{\delta\mathcal{L}}{\delta T} = \int_0^\infty e^{-\rho t}\left[\left\langle\frac{\partial\zeta_{i,t}}{\partial b}, v_t f_{i,t}\right\rangle + \left\langle\varrho_{i,t}, v_t\frac{\partial\mathcal{V}_{i,t}}{\partial b}\right\rangle + \eta_{T,t}v_t\right]dt = 0 \tag{205}$$

$$\zeta_{i,t}f_{i,t}\Big|_{b=\underline{b}} = 0 \tag{206}$$

Strong form:

$$\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{T,t} = 0 \tag{207}$$

7.6.10 Control over dividends

Weak form:

$$\frac{\delta \mathcal{L}}{\delta d} = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, v_t \varepsilon_{i,t} f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, v_t \varepsilon_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{d,t} v_t \right] dt = 0 \tag{208}$$

$$\zeta_{i,t} f_{i,t} \Big|_{b=\underline{b}} = 0 \tag{209}$$

Strong form:

$$\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, \varepsilon_{i,t} f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, \varepsilon_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{d,t} = 0 \tag{210}$$